

OPTIMAL RESULTS FOR THE FRACTIONAL HEAT EQUATION INVOLVING THE HARDY POTENTIAL

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ABSTRACT. In this paper we study the influence of the Hardy potential in the fractional heat equation. In particular, we consider the problem

$$(P_\theta) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + \theta u^p + cf \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

where $N > 2s$, $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian of order $2s$, $p > 1$, $c, \lambda > 0$, $\theta = \{0, 1\}$, and $u_0, f \geq 0$ are in a suitable class of functions.

The main results in the article are:

- (1) Optimal results about *existence* and *instantaneous and complete blow up* in the linear problem (P_0) , where the best constant in the fractional Hardy inequality, $\Lambda_{N,s}$, provides the threshold between existence and nonexistence. To obtain local sharp estimates of the solutions it is required to prove a weak Harnack inequality for a weighted operator that appears in a natural way.
- (2) The existence of a critical power $p_+(s, \lambda)$ in the semilinear problem (P_1) such that:
 - (a) If $p > p_+(s, \lambda)$, the problem has no weak positive supersolutions and a phenomenon of *complete and instantaneous blow up* happens.
 - (b) If $p < p_+(s, \lambda)$, there exists a positive solution for a suitable class of nonnegative data.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this work we will study the solvability of the following linear problem,

$$(1) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + f \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

and of the semilinear problem,

$$(2) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p + f \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

where Ω is a $C^{1,1}$ bounded domain in \mathbb{R}^N , $N > 2s$, $0 < s < 1$, $p > 1$, and c and λ are positive constants. To avoid the trivial case, we assume $0 \in \Omega$. We suppose that f and u_0 are non negative

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functions satisfying some hypotheses that we will precise later. By $(-\Delta)^s$ we denote the fractional Laplacian of order $2s$, that is,

$$(3) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad s \in (0, 1),$$

where

$$a_{N,s} := 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$$

is the normalization constant so that the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u), \quad \xi \in \mathbb{R}^N, s \in (0, 1),$$

holds for every $u \in \mathcal{S}(\mathbb{R}^N)$, the Schwartz class (see [27]). This last identity justifies why we call fractional Laplacian to the integral operator. Notice that $(-\Delta)^s u$ is well defined if, for instance, $u \in \mathcal{L}^s(\mathbb{R}^N) \cap C_{\text{loc}}^{2s+\beta}(\mathbb{R}^N)$ (or $C^{1,2s+\beta-1}$ if $2s + \beta > 1$), for some $\beta > 0$. Here

$$\mathcal{L}^s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} dx < +\infty \right\},$$

endowed with the norm

$$\|u\|_{\mathcal{L}^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} dx.$$

Notice also that, for u under these hypotheses, $(-\Delta)^s u$ is a continuous function. See [41].

In the case $s = 1$, these problems correspond to the classical heat equation, and they have been deeply understood in the past years (see for instance [9] for (1) and [3] for (2)). For $s \in (0, 1)$, the fractional setting, there exists also a large literature dealing with the case $\lambda = 0$. We refer for instance to [15, 24, 31] and the references therein. A result on the uniqueness of positive regular solution to the linear problem can be found in [11].

However, the case under consideration here, $\lambda > 0$ and $s \in (0, 1)$, is quite different; for instance, any positive supersolution to problem (1) is unbounded close to the origin, even for nice data. This fact, among other results, was proved in the local case by Baras-Goldstein in [9]. In the nonlocal framework, the precise rate of growth of the solutions near the origin will be the key to obtain the optimal results. It is worthy to point out here the difference with the local case, where this rate is obtained just by solving an elementary linear differential equation.

The main results in this work can be summarized as follows.

First, to study the local behavior of the solutions we need some sharp local estimates, that are based on a *Harnack inequality* for a related problem resulting from the *ground state transformation* by Frank, Lieb and Seiringer ([27]), which is a problem with *singular coefficients*. This weak Harnack inequality gives the exact blow up rate for the positive supersolutions near the t axis. For the proof of this result, we closely follow the work of Felsinger and Kassmann in [24], where the authors develop a weak parabolic Harnack inequality for a general type of nonlocal operators. This result does not apply straightforward to our singular operator, so, based on their scheme, we need to check every step in the Moser's protocol (see [37]). The Harnack inequality for singular weights can be useful in related problems.

The results obtained in this work for the linear problem (1) can be seen as the extension to the fractional setting of those for the heat equation developed by P. Baras and J. A. Goldstein in [9]. Nevertheless the proofs that we present are significantly different. More precisely, we obtain the optimal summability required to the data in order to solve the problem and to prove the instantaneous and complete blow up for $\lambda > \Lambda_{N,s}$ by using the results of Section 4. As a byproduct, we also prove the optimality of the power in the Hardy potential term, i.e., that $p > 1$ is a *supercritical power* for $\frac{u^p}{|x|^{2s}}$. This result in the local framework was obtained by Brezis and Cabré in [14].

Secondly, concerning the semilinear problem (2), the main result that we obtain in this paper is that for all $0 < \lambda < \Lambda_{N,s}$ there exists a threshold exponent, $p_+(\lambda, s)$, for the existence of positive solutions. By *threshold* we mean that when we consider an exponent $p > p_+(\lambda, s)$, there are no positive supersolutions even in the weak sense, while if $p < p_+(s, \lambda)$, it is possible to establish a suitable class of nonnegative data for which we can find a positive solution. We will see in particular that the threshold exponent $p_+(s, \lambda)$ is the same as in the elliptic case (see [10, 23]). In fact, this critical power is related to the possibility of finding a supersolution to the elliptic problem in the whole \mathbb{R}^N . As in the linear problem the main ingredient is the local estimates of the solutions close to the origin.

The paper is organized as follows.

In Section 2 we describe the natural functional framework associated to the problems (1) and (2). We define the two notions of solution we will use along the paper: *weak* solutions and *energy* solutions. Moreover, we prove some comparison principles which are interesting themselves.

In Section 3 we describe the radial solutions of the corresponding homogeneous elliptic problem in \mathbb{R}^N . These solutions will allow us to precise the singularity of the supersolutions to problem (1) near the origin. This is a key point to perform the so called *ground state transformation* (see [27]), and to obtain the nonexistence results afterwards.

Section 4 is devoted to obtain the weak Harnack inequality for the positive supersolutions of the problem resulting from the *ground state transformation* by Frank, Lieb and Seiringer, that introduces the difficulty of dealing with a kernel with *singular coefficients*.

The goal of Section 5 is to study the linear problem (1) and some consequences.

Finally, in Section 6 we consider the semilinear problem (2). Furthermore, we prove a instantaneous and complete blow-up phenomenon.

We also include two appendices with auxiliary results that seem to be useful in other applications.

Appendix A includes a Hölder regularity result, that can be seen as the translation to the bounded domain case of some results in [15]. More precisely, we will prove that an energy solution is in fact a viscosity solution, and we can apply regularity results that ensure that it is indeed a *strong solution* in the sense of Definition 1.3 in [11].

In Appendix B we include some inequalities needed to prove the Harnack inequality in Section 4. As far as we know, these results involve some significative changes respect to the standard ones, and therefore we consider them appropriate to be included here.

2. FUNCTIONAL FRAMEWORK: SOME PRELIMINARY RESULTS

Along this paper we will always assume that Ω is a $C^{1,1}$ bounded domain of \mathbb{R}^N with the origin inside. We consider the fractional Sobolev space $H_0^s(\Omega)$ defined as

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm

$$\|u\|_{H_0^s(\Omega)} := \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2},$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. The pair $(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ yields a Hilbert space. Moreover,

$$(-\Delta)^s : H_0^s(\Omega) \rightarrow H^{-s}(\Omega),$$

is a continuous operator, where $(-\Delta)^s$ is defined in (3).

In what follows we will use the relation between the norm in the space $H_0^s(\Omega)$ and the L^2 norm of the fractional Laplacian, see [21, Proposition 3.6],

$$(4) \quad \|u\|_{H_0^s(\Omega)}^2 = 2a_{N,s}^{-1} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2.$$

It is easy to check that for u and φ smooth enough, with vanishing conditions outside Ω , we have the following duality product,

$$2a_{N,s}^{-1} \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = \int_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy,$$

that in particular implies the selfadjointness of $(-\Delta)^s$ in $H_0^s(\Omega)$.

We enunciate a Sobolev-type inequality that we will use throughout the paper (see for example [21] for a proof).

Theorem 2.1. (*Sobolev embedding*). *Let $s \in (0, 1)$. There exists a constant $S = S(N, s)$ such that, for all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have*

$$\|\phi\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq S \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,$$

being

$$2_s^* = \frac{2N}{N - 2s},$$

the so called fractional critical exponent.

The parabolic problems studied in this article are related to the following Hardy inequality, proved in [30] (see also [12, 27, 43, 45]).

Theorem 2.2. (*Fractional Hardy inequality*). *For all $u \in C_0^\infty(\mathbb{R}^N)$ the following inequality holds,*

$$(5) \quad \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 \, d\xi \geq \Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} u^2 \, dx,$$

where

$$(6) \quad \Lambda_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

The constant $\Lambda_{N,s}$ is optimal and not attained.

Remark 2.3.

- (1) It can be checked that

$$\Lambda_{N,s} \rightarrow \Lambda_{N,1} := \left(\frac{N-2}{2} \right)^2,$$

the classical Hardy constant, when s tends to 1. Moreover, by scaling it can be proved that the optimal constant is the same for every domain containing the pole of the Hardy potential.

- (2) The optimal constant defined in (6) coincides for every bounded domain Ω containing the pole of the Hardy potential. That is, if $0 \in \Omega$, using (4) we can rewrite the Hardy inequality (5) as

$$(7) \quad \frac{a_{N,s}}{2} \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \geq \Lambda_{N,s} \int_\Omega \frac{u^2}{|x|^{2s}} \, dx, \quad u \in H_0^s(\Omega).$$

The optimality of $\Lambda_{N,s}$ here follows by a scaling argument.

Consider the parabolic problem

$$(P) := \begin{cases} u_t + (-\Delta)^s u &= f(x, t, u) \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

Denote

$$\begin{aligned} \mathcal{T} := \{ \phi : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}, \text{ s.t. } -\phi_t + (-\Delta)^s \phi = \varphi, \varphi \in L^\infty(\Omega \times (0, T)) \cap \mathcal{C}^{\alpha, \beta}(\Omega \times (0, T)), \\ \phi = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0, T], \phi(x, T) = 0 \text{ in } \Omega \}. \end{aligned}$$

Notice that every $\phi \in \mathcal{T}$ belongs in particular to $L^\infty(\Omega \times (0, T))$ (see [34]). Moreover according with the results in Appendix A, $\phi \in \mathcal{T}$ is a strong solution (see Definition 1.3 in [11]).

We define the meaning of *weak solution*.

Definition 2.4. Assume $u_0 \in L^1(\Omega)$. We say that $u \in \mathcal{C}([0, T]; L^1(\Omega))$, is a weak supersolution (subsolution) of problem (P) if $f(x, t, u) \in L^1(\Omega \times [0, T])$, $u \geq (\leq) 0$ in $(\mathbb{R}^N \setminus \Omega) \times [0, T]$, $u(x, 0) \geq (\leq) u_0(x)$ in Ω , and for all nonnegative $\phi \in \mathcal{T}$ we have that

$$(8) \quad \int_0^T \int_\Omega -\phi_t u \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} u (-\Delta)^s \phi \, dx \, dt \geq (\leq) \int_0^T \int_\Omega f \phi \, dx \, dt + \int_\Omega u_0(x) \phi(x, 0) \, dx.$$

If u is super and subsolution then we say that u is a weak solution to (P).

The weak solution will be considered to formulate the optimal nonexistence results. For existence results, we will consider the classical notion of *finite energy solutions*.

Definition 2.5. Assume $u_0(x) \in L^2(\Omega)$. We say that $u \in L^2(0, T; H^s(\mathbb{R}^N))$ with $u_t \in L^2(0, T; H^{-s}(\Omega))$ is a finite energy supersolution (respectively subsolution) of (P) if $f(x, t, u) \in L^2(0, T; H^{-s}(\Omega))$ and it satisfies

$$\begin{aligned} \int_0^T \int_\Omega u_t \varphi \, dx \, dt + \frac{a_{N,s}}{2} \int_0^T \int_Q \frac{(u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t))}{|x - y|^{N+2s}} \, dx \, dy \, dt \\ \geq (\leq) \int_0^T \int_\Omega f \varphi \, dx \, dt, \end{aligned}$$

for any nonnegative $\varphi \in L^2(0, T; H_0^s(\Omega))$, $\varphi = 0$ in $(\mathbb{R}^N \setminus \Omega) \times (0, T)$.

If u is super and subsolution then u is a finite energy solution.

Remark 2.6. If $u \in L^2(0, T; H_0^s(\Omega))$ and $u_t \in L^2(0, T; H^{-s}(\Omega))$, then by approximating with smooth functions and taking advantage of the hilbertian structure of the space, it can be checked that $u \in \mathcal{C}([0, T]; L^2(\Omega))$.

Notice that both definitions can be considered, by scaling, in $\mathbb{R}^N \times [T_1, T_2]$ with $[T_1, T_2] \subset [0, T]$.

The existence and uniqueness of an energy solution to problem (P) when F is in the dual space $L^2(0, T; H^{-s}(\Omega))$ can be obtained by means of a direct Hilbert space approach. See the result by A. N. Milgram in [36] based on a method of Vishik in [44], that is essentially an extension of the Lax-Milgram theorem to parabolic problems. More precisely, we have the following result.

Theorem 2.7. *Let $f \in L^2(0, T; H^{-s}(\Omega))$, then problem (1) has a unique finite energy solution.*

See [34, Theorem 26] for a detailed proof in this fractional framework.

Remark 2.8. Notice that by defining

$$\begin{aligned} L_\phi(u) &:= \int_0^T \int_\Omega -u \phi_t \, dx \, dt + \frac{a_{N,s}}{2} \int_0^T \int_Q \frac{(u(x, t) - u(y, t))(\phi(x, t) - \phi(y, t))}{|x - y|^{N+2s}} \, dx \, dy \, dt \\ (9) \quad &- \lambda \int_0^T \int_\Omega \frac{u \phi}{|x|^{2s}} \, dx \, dt, \end{aligned}$$

and

$$\langle \varphi, \phi \rangle_* = \frac{1}{2} \langle \varphi(x, 0), \phi(x, 0) \rangle_{L^2(\Omega)} + \frac{a_{N,s}}{2} \left(1 - \frac{\lambda}{\Lambda_{N,s}} \right) \langle \varphi, \phi \rangle_{L^2(0, T; H_0^s(\Omega))},$$

thanks to the Hardy inequality (see (5)) one can reproduce the proof of [34, Theorem 26] to assure the existence and uniqueness of an energy solution to the problem

$$(P_\lambda) := \begin{cases} u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} &= F(x, t) \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

for $F \in L^2(0, T; H^{-s}(\Omega))$, $u_0 \in L^2(\Omega)$, and $\lambda < \Lambda_{N,s}$. For the case $\lambda = \Lambda_{N,s}$, consider the Hilbert space $H(\Omega)$ defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$(10) \quad \|u\|_{H(\Omega)}^2 := \frac{a_{N,s}}{2} \|u\|_{H_0^s(\Omega)}^2 - \Lambda_{N,s} \int_{\Omega} \frac{u^2}{|x|^{2s}} dx.$$

In [28], the author proves the following improved Hardy inequality,

$$(11) \quad \frac{a_{N,s}}{2} \|u\|_{H_0^s(\Omega)}^2 - \Lambda_{N,s} \int_{\Omega} \frac{u^2}{|x|^{2s}} dx \geq C(\Omega, q, N, s) \|u\|_{W_0^{\tau,2}(\Omega)}^2,$$

for all $s/2 < \tau < s$ (see also [23, 4] for alternative proofs without using the Fourier transform). Thus we can see that $H(\Omega) \subset W_0^{\tau,2}(\Omega)$ and therefore, $H(\Omega)$ is compactly embedded in $L^p(\Omega)$ for all $1 \leq p < 2_s^*$ (see [21, Corollary 7.2]). Therefore, the proof remains the same considering $L_\phi(u)$ as in (9) (setting $\lambda = \Lambda_{N,s}$), and defining the scalar product $\langle \cdot, \cdot \rangle_*$ as

$$\langle \varphi, \phi \rangle_* = \frac{1}{2} \langle \varphi(x, 0), \phi(x, 0) \rangle_{L^2(\Omega)} + \langle \varphi, \phi \rangle_{L^2(0,T;H_0^s(\Omega))},$$

where the last term follows from (10).

In order to study monotonicity approaches, we will need to prove comparison results for both kind of solutions.

Lemma 2.9. (*Weak Comparison Principle*). *Let $0 \leq \lambda \leq \Lambda_{N,s}$ and let $u, v \in \mathcal{C}([T_1, T_2]; L^1(\Omega))$ be weak solutions to the problems*

$$\begin{cases} u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f_1 \text{ in } \Omega \times (T_1, T_2), \\ u = g_1 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ u(x, T_1) = h_1(x) \text{ in } \Omega, \end{cases} \quad \begin{cases} v_t + (-\Delta)^s v - \lambda \frac{v}{|x|^{2s}} = f_2 \text{ in } \Omega \times (T_1, T_2), \\ v = g_2 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ v(x, T_1) = h_2(x) \text{ in } \Omega, \end{cases}$$

respectively, where $f_1, f_2 \in L^1(\Omega \times (T_1, T_2))$, $g_1, g_2 \in L^1((\mathbb{R}^N \setminus \Omega) \times (T_1, T_2))$ and $h_1, h_2 \in L^1(\Omega)$.

If $f_1 \leq f_2$ in $\Omega \times (T_1, T_2)$, $g_1 \leq g_2$ in $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2)$ and $h_1 \leq h_2$ in Ω , then $u \leq v$ in $\mathbb{R}^N \times (T_1, T_2)$.

Proof. Define $w = v - u$. Hence, w is a weak solution of

$$\begin{cases} w_t + (-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = f_2 - f_1 \geq 0 \text{ in } \Omega \times (T_1, T_2), \\ w = g_2 - g_1 \geq 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ w(x, T_1) = h_2 - h_1 \geq 0 \text{ in } \Omega. \end{cases}$$

Consider now $\Phi \in \mathcal{C}_0^\infty(\Omega \times (T_1, T_2))$, $\Phi \geq 0$, and the solution φ_n to the problem

$$(12) \quad \begin{cases} -(\varphi_n)_t + (-\Delta)^s \varphi_n &= \lambda \frac{\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} + \Phi \text{ in } \Omega \times (T_1, T_2), \\ \varphi_n &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ \varphi_n(x, T_2) &= 0 \text{ in } \Omega, \end{cases}$$

with

$$(13) \quad \begin{cases} -(\varphi_0)_t + (-\Delta)^s \varphi_0 &= \Phi & \text{in } \Omega \times (T_1, T_2), \\ \varphi_0 &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2], \\ \varphi_0(x, T_2) &= 0 & \text{in } \Omega. \end{cases}$$

Since φ_n is regular in $\Omega \times [T_1, T_2)$ and bounded in $\mathbb{R}^N \times (T_1, T_2)$ (see Appendix A), this equation can be understood in a pointwise sense. Moreover, by the Strong Comparison Principle, we know that $\varphi_n \geq 0$ and $\varphi_{n-1} \leq \varphi_n$ in $\mathbb{R}^N \times [T_1, T_2)$ for all $n \in \mathbb{N}$.

Hence, by the definition of weak solutions, and using that $w \geq 0$

$$\begin{aligned} \int_{T_1}^{T_2} \int_{\Omega} w \Phi \, dx \, dt &= \int_{T_1}^{T_2} \int_{\Omega} w(-\varphi_n)_t \, dx \, dt + \int_{T_1}^{T_2} \int_{\Omega} w(-\Delta)^s \varphi_n \, dx \, dt - \lambda \int_{T_1}^{T_2} \int_{\Omega} \frac{w \varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} \, dx \, dt \\ &\geq \int_{T_1}^{T_2} \int_{\Omega} w(-\varphi_n)_t \, dx \, dt + \int_{T_1}^{T_2} \int_{\Omega} w(-\Delta)^s \varphi_n \, dx \, dt - \lambda \int_{T_1}^{T_2} \int_{\Omega} \frac{w \varphi_n}{|x|^{2s}} \, dx \, dt \\ &= \int_{T_1}^{T_2} \int_{\Omega} (f_2 - f_1) \varphi_n \, dx \, dt + \int_{\Omega} w(x, T_1) \varphi_n(x, T_1) \, dx \geq 0, \end{aligned}$$

for all $\Phi \in C_0^\infty(\Omega \times (T_1, T_2))$, $\Phi \geq 0$. Thus, $w \geq 0$ in $\mathbb{R}^N \times (T_1, T_2)$, and therefore $u \leq v$ in $\mathbb{R}^N \times (T_1, T_2)$. \square

Corollary 2.10. (*Uniqueness of weak solutions for the linear problem*).

Let suppose $F \in L^1(\Omega \times (0, T))$. Then problem (P_λ) has at most one nontrivial weak solution.

The comparison result for energy solutions can be proved in a standard way, so we skip the proof (see for example [10] for a proof in the elliptic case).

Lemma 2.11. *Energy Comparison Principle. Let $0 \leq \lambda < \Lambda_{N,s}$ and let $u, v \in L^2(T_1, T_2; H^s(\mathbb{R}^N))$ with $u_t, v_t \in L^2(T_1, T_2; H^{-s}(\Omega))$ be finite energy solutions to the problems*

$$\begin{cases} u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f_1 & \text{in } \Omega \times (T_1, T_2), \\ u = g_1 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2], \\ u(x, T_1) = h_1(x) & \text{in } \Omega, \end{cases} \quad \begin{cases} v_t + (-\Delta)^s v - \lambda \frac{v}{|x|^{2s}} = f_2 & \text{in } \Omega \times (T_1, T_2), \\ v = g_2 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2], \\ v(x, T_1) = h_2(x) & \text{in } \Omega, \end{cases}$$

respectively, where $f_1, f_2 \in L^2(T_1, T_2; H^{-s}(\Omega))$, $g_1, g_2 \in L^2(T_1, T_2; L^2(\mathbb{R}^N \setminus \Omega))$ and $h_1, h_2 \in L^2(\Omega)$. If $f_1 \leq f_2$ in $\Omega \times (T_1, T_2)$, $g_1 \leq g_2$ in $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2]$ and $h_1 \leq h_2$ in Ω , then $u \leq v$ in $\mathbb{R}^N \times (T_1, T_2)$.

Remark 2.12. Notice that if $\lambda = \Lambda_{N,s}$, we can obtain the same result for $u, v \in L^2(T_1, T_2; H(\Omega))$, where $H(\Omega)$ was defined in (10), only by exactly repeating this proof.

Finally, consider the problem

$$(14) \quad \begin{cases} u_t + (-\Delta)^s u &= 0 & \text{in } \Omega \times (0, T), \\ u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \geq 0 & \text{if } x \in \Omega. \end{cases}$$

We enunciate a *Weak Harnack Inequality* that we will use along the paper (see [24, Theorem 1.1] even in a more general setting).

Lemma 2.13. (*Weak Harnack Inequality*). If u is a non negative supersolution of (14) in $\Omega \times (0, T)$, then for every $t_0 \in (0, T)$ there exists $r > 0$ and a positive constant $C = C(N, s, r, t_0, \beta)$ such that

$$\iint_{R^-} u(x, t) \, dx \, dt \leq C (\text{ess inf}_{R^+} u),$$

where $R^- = B_r(0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$, $R^+ = B_r(0) \times (t_0 + \frac{1}{4}\beta, t_0 + \frac{3}{4}\beta)$.

As a consequence of this lemma, we can formulate the *strong maximum principle*.

Theorem 2.14. (*Strong Maximum Principle*). *If u is a non negative supersolution of (14), then $u(x, t) > 0$ in $\Omega \times (0, T)$.*

3. LOCAL BEHAVIOR OF SOLUTIONS OF THE STATIONARY EQUATION

The purpose of this section is to analyze the behavior of the radial solutions to the homogeneous problem

$$(15) \quad (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \text{ in } \mathbb{R}^N \setminus \{0\}$$

in a neighborhood of the origin, in order to use this information as a tool for proving the existence and nonexistence results.

Lemma 3.1. *Let $0 < \lambda \leq \Lambda_{N,s}$. Then $v_{\pm\alpha} = |x|^{-\frac{N-2s}{2} \pm \alpha}$ are solutions to*

$$(16) \quad (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \text{ in } (\mathbb{R}^N \setminus \{0\}),$$

where α is obtained by the identity

$$(17) \quad \lambda = \lambda(\alpha) = \lambda(-\alpha) = \frac{2^{2s} \Gamma(\frac{N+2s+2\alpha}{4}) \Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4}) \Gamma(\frac{N-2s-2\alpha}{4})}.$$

Proof. Applying the Fourier transform of radial functions (see for instance [43, Theorem 4.1]) it yields,

$$\begin{aligned} \mathcal{F}(v_\alpha)(\xi) &= \xi^{-\frac{N-1}{2}} \int_0^\infty (r\xi)^{\frac{1}{2}} J_{\frac{N-2}{2}}(r\xi) v_\alpha(r) r^{\frac{N-1}{2}} dr = \xi^{-\frac{N}{2}-s-\alpha} \int_0^\infty (r\xi)^{s+\alpha} J_{\frac{N-2}{2}}(r\xi) d(r\xi) \\ &= 2^{\alpha+s} \frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})} \xi^{-\frac{N}{2}-s-\alpha}, \end{aligned}$$

where $J_{\frac{N-2}{2}}$ denotes the Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}.$$

Now, we notice that

$$(-\Delta)^s v_\alpha = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(v_\alpha)(\xi)) = 2^{\alpha+s} \frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})} \mathcal{F}^{-1}(|\xi|^{-\frac{N}{2}+s-\alpha}) = \lambda |x|^{-2s} v_\alpha$$

with $\lambda = \lambda(\alpha)$ equal to (17). □

Remark 3.2. Notice that $\lambda(\alpha) = \lambda(-\alpha) = m_\alpha m_{-\alpha}$, with $m_\alpha = 2^{\alpha+s} \frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})}$.

Lemma 3.3. *The following equivalence holds true:*

$$0 < \lambda(\alpha) = \lambda(-\alpha) \leq \Lambda_{N,s} \text{ if and only if } 0 \leq \alpha < \frac{N-2s}{2}.$$

For the reader convenience, we include an elemental proof of this Lemma (see also [27, 30]).

Proof. Notice that $\lambda(\alpha)$ is a positive continuous function for $0 \leq \alpha < \frac{N-2s}{2}$, such that $\lambda(0) = \Lambda_{N,s}$. It is sufficient to prove that for fixed s , $\lambda(\alpha)$ is a decreasing function.

Let consider the following representation of the Gamma function (see [7] for more details):

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}},$$

where γ is the Euler-Mascheroni constant (see for instance [5]). We aim to prove that $\log \frac{1}{\lambda(\alpha)}$ is an increasing function in α .

$$\begin{aligned} \log \frac{1}{\lambda(\alpha)} &= \log \frac{1}{2^{2s}} \frac{\frac{1}{\Gamma(\frac{N+2s+2\alpha}{4})} \cdot \frac{1}{\Gamma(\frac{N+2s-2\alpha}{4})}}{\frac{1}{\Gamma(\frac{N-2s+2\alpha}{4})} \cdot \frac{1}{\Gamma(\frac{N-2s-2\alpha}{4})}} \\ &= -2s \cdot \log 2 + \log \frac{(N+2s)^2 - 4\alpha^2}{(N-2s)^2 - 4\alpha^2} + 2\gamma s + \sum_{n=1}^{\infty} \left[\log \frac{(\frac{N+4n+2s}{4n})^2 - \frac{\alpha^2}{4n^2}}{(\frac{N+4n-2s}{4n})^2 - \frac{\alpha^2}{4n^2}} - \frac{2s}{n} \right]. \end{aligned}$$

Notice that the last term is a convergent series in the same way as

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

is a convergent product. We conclude just by noticing that if $a > b$ and $\zeta > 0$, then $\frac{a^2 - \zeta^2}{b^2 - \zeta^2}$ is an increasing function in ζ . \square

Remark 3.4. Notice that we can explicitly construct two positive solutions to the homogeneous problem (16). Henceforth, we denote

$$(18) \quad \gamma = \frac{N-2s}{2} - \alpha \text{ and } \bar{\gamma} = \frac{N-2s}{2} + \alpha,$$

with $0 < \gamma \leq \frac{N-2s}{2} \leq \bar{\gamma} < (N-2s)$. Since $N-2\gamma-2s = 2\alpha > 0$ and $N-2\bar{\gamma}-2s = -2\alpha < 0$, then $(-\Delta)^{s/2}(|x|^{-\gamma}) \in L^2(\Omega)$, but $(-\Delta)^{s/2}(|x|^{-\bar{\gamma}})$ does not.

We will use these results to study the unboundedness of any positive weak supersolution, and moreover, to obtain an explicit quantitative information on the growth around the origin when the summability of the datum is good enough.

4. WEAK HARNACK INEQUALITY FOR A WEIGHTED PROBLEM

Frank, Lieb and Seiringer proved in [27, Proposition 4.1] the following representation result.

Lemma 4.1. (*Ground State Representation*) Let $0 < \gamma < \frac{N-2s}{2}$. If $\phi \in C_0^\infty(\mathbb{R}^N)$ and $\bar{\phi}(x) := |x|^\gamma \phi(x)$, then

$$(19) \quad \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{\phi}(\xi)|^2 d\xi - (\Lambda_{N,s} + \Phi_{N,s}(\gamma)) \int_{\mathbb{R}^N} |x|^{-2s} |\phi(x)|^2 dx = \frac{a_{N,s}}{2} \int \int_{\mathbb{R}^{2N}} \frac{|\bar{\phi}(x) - \bar{\phi}(y)|^2}{|x-y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma},$$

where

$$(20) \quad \Phi_{N,s}(\gamma) := 2^{2s} \left(\frac{\Gamma(\frac{\gamma+2s}{2}) \Gamma(\frac{N-\gamma}{2})}{\Gamma(\frac{N-\gamma-2s}{2}) \Gamma(\frac{\gamma}{2})} - \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} \right).$$

A relevant fact for us is the following result.

Proposition 4.2. *Consider the function*

$$\Psi_{N,s} : [0, \frac{N-2s}{2}] \rightarrow [0, \Lambda_{N,s}]$$

$$\gamma \rightarrow \Psi_{N,s}(\gamma) := \Lambda_{N,s} + \Phi_{N,s}(\gamma),$$

where $\Phi_{N,s}$ is defined by (20). Then $\Psi_{N,s}$ is strictly increasing and surjective.

Notice that, considering γ defined in (18), $\lambda(\alpha) = \Psi_{N,s}\left(\frac{N-2s}{2} - \alpha\right)$, and therefore for any $0 < \lambda < \Lambda_{N,s}$, there exists $\alpha \in (0, \frac{N-2s}{2})$, such that

$$\lambda = \lambda(\alpha) = \frac{2^{2s} \Gamma(\frac{N+2s+2\alpha}{4}) \Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4}) \Gamma(\frac{N-2s-2\alpha}{4})}.$$

Taking $0 < \gamma = \frac{N-2s}{2} - \alpha < \frac{N-2s}{2}$, and $\Lambda_{N,s} + \Phi_{N,s}(\gamma) = \lambda(\alpha)$, by Lemma 4.1 we can write the energy as

$$(21) \quad \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi, t)|^2 d\xi - \lambda(\alpha) \int_{\mathbb{R}^N} |x|^{-2s} |u(x, t)|^2 dx = \frac{a_{N,s}}{2} \int \int_{\mathbb{R}^{2N}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma},$$

where $v(x, t) := |x|^\gamma u(x, t)$. The Euler-Lagrange equation associated to this identity is

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = |x|^\gamma L_\gamma v(x, t),$$

where

$$L_\gamma(v(x, t)) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} (v(x, t) - v(y, t)) K(x, y) dy,$$

and

$$K(x, y) = \frac{1}{|x|^\gamma} \frac{1}{|y|^\gamma} \frac{1}{|x - y|^{N+2s}}, \quad 0 < \gamma = \frac{N-2s}{2} - \alpha < \frac{N-2s}{2}.$$

Thus we conclude that if u is an energy solution of problem (1) with $0 < \lambda < \Lambda_{N,s}$, then v solves the parabolic equation

$$(22) \quad \begin{cases} |x|^{-2\gamma} v_t + L_\gamma v &= |x|^{-\gamma} f(x, t) \text{ in } \Omega \times (0, T), \\ v(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ v(x, 0) &= v_0(x) := |x|^\gamma u_0(x) \text{ if } x \in \Omega. \end{cases}$$

Therefore, if we want to analyze the behavior of u near the origin, we may, equivalently, deal with the same question for v . In particular, we will prove that the weighted operator

$$|x|^{-2\gamma} v_t - L_\gamma v,$$

satisfies a suitable weak Harnack inequality. In the local case, this kind of result can be obtained as a consequence of some results by Chiarenza-Frasca, Chiarenza-Serapioni and Gutierrez-Wheden, see [18, 19, 29] and the references therein.

Let us precise first the natural functional framework associated to the new problem (22). For simplicity of typing we denote

$$(23) \quad d\mu := \frac{dx}{|x|^{2\gamma}}, \quad \text{and} \quad d\nu := K(x, y) dx dy.$$

Let $\Omega \subseteq \mathbb{R}^N$. We define the weighted Sobolev space $Y^{s,\gamma}(\Omega)$ as

$$Y^{s,\gamma}(\Omega) := \left\{ \phi \in L^2(\Omega, d\mu) : \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu < +\infty \right\}.$$

It is clear that $Y^{s,\gamma}(\Omega)$ is a Hilbert space endowed with the norm

$$\|\phi\|_{Y^{s,\gamma}(\Omega)} := \left(\int_{\Omega} |\phi(x)|^2 d\mu + \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu \right)^{\frac{1}{2}},$$

and we define the space $Y_0^{s,\gamma}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to this norm. In particular, we denote

$$|||\phi|||_{Y_0^{s,\gamma}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu \right)^{\frac{1}{2}}.$$

If Ω is bounded, the norms $|||\cdot|||_{Y_0^{s,\gamma}(\Omega)}$ and $\|\cdot\|_{Y^{s,\gamma}(\Omega)}$ are equivalent (see Theorem B.2 in Appendix B for more details). If $\Omega = \mathbb{R}^N$, using the definition of L_γ , we obtain that for all $w_1, w_2 \in Y_0^{s,\gamma}(\mathbb{R}^N)$,

$$\langle L_\gamma(w_1), w_2 \rangle = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (w_1(x) - w_1(y))(w_2(x) - w_2(y)) d\nu.$$

Let us begin by the following natural definition.

Definition 4.3. Let $v \in C([0, T], L^2(\mathbb{R}^N, d\mu)) \cap L^2(0, T; Y^{s,\gamma}(\mathbb{R}^N))$. We say that v is a supersolution to problem (22) if $v(x, 0) \geq v_0(x)$ and for all $\Omega_1 \subset \subset \Omega$, for all $[t_1, t_2] \subset (0, T)$ we have

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega_1} -\varphi_t v d\mu dt + \frac{a_{N,s}}{2} \int_{t_1}^{t_2} \int_{\tilde{Q}} (v(x, t) - v(y, t))(\varphi(x, t) - \varphi(y, t)) d\nu dt \\ & \geq \int_{t_1}^{t_2} \int_{\Omega_1} f \varphi |x|^{-\gamma} dx dt + \int_{\Omega_1} \varphi(x, t_1) v(x, t_1) d\mu - \int_{\Omega_1} \varphi(x, t_2) v(x, t_2) d\mu, \end{aligned}$$

for any nonnegative $\varphi \in L^2(t_1, t_2; Y_0^{s,\gamma}(\Omega_1))$, such that $\varphi_t \in L^2(t_1, t_2; Y^{-s,\gamma}(\Omega_1))$, where $\tilde{Q} := \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega_1 \times \mathcal{C}\Omega_1)$.

The main result of this section is the next Theorem.

Theorem 4.4. (*Weak Harnack Inequality*) Assume that f (resp. v_0) ≥ 0 in $\Omega \times (0, T)$ (resp. in Ω). Let $v \in C([0, T], L^2(\mathbb{R}^N, d\mu)) \cap L^2(0, T; Y^{s,\gamma}(\mathbb{R}^N))$ be a supersolution to (22) with $v \not\equiv 0$ in $\mathbb{R}^N \times (0, T)$.

Then for any $q < 1 + \frac{2s}{N}$, we have

$$(24) \quad \left(\iint_{Q_1} v^q d\mu dt \right)^{\frac{1}{q}} \leq C \inf_{Q_2} v,$$

where $Q_1 = B_r(x_0) \times (t_1, t_2)$, $Q_2 = B_r(x_0) \times (t_3, t_4)$ with $0 < t_1 < t_2 < t_3 < t_4 < T$, $B_r(x_0) \subset \Omega$ and $C = C(N, r, t_1, t_2, t_3, t_4) > 0$.

The proof of this result follows the classical arguments by Moser (see [37]) with some necessary adaptations. In the context of the parabolic fractional-like operators the precedent work is the interesting paper by Felsinger and Kassmann, [24], that we will closely follow here, adapting the proofs to the weighted operator appearing in (22). To make the paper self-contained, we include here the corresponding proofs.

First of all, we will need an iteration result, originally proved in [37] and extended by Bombieri and Giusti in [13] to the case of general measures in the elliptic setting (see also [40, Lemma 2.2.6]).

Lemma 4.5. Let $\{U(r)\}_{\theta \leq r \leq 1}$ be a nondecreasing family of bounded domains $U(r) \subset \mathbb{R}^{N+1}$, and let m, c_0 be positive constants, $\eta \in (0, 1)$, $\theta \in [\frac{1}{2}, 1]$ and $0 < p_0 \leq +\infty$. Let w be a positive, measurable function defined on $U(1)$ satisfying

$$\|w\|_{L^{p_0}(U(r), d\mu dt)} \leq \left(\frac{c_0}{(R-r)^m |U(1)|_{d\mu \times dt}} \right)^{\left(\frac{1}{p_0} - \frac{1}{p} \right)} \left(\iint_{U(R)} w^p d\mu dt \right)^{\frac{1}{p}}$$

for all $r, R \in [\theta, 1]$, $r < R$, and for all $p \in (0, \min\{\eta p_0, 1\})$.

Assume also that

$$\forall s > 0 : |U(1) \cap \{\log w > s\}|_{d\mu \times dt} \leq \frac{c_0 |U(1)|_{d\mu \times dt}}{s}.$$

Then there exists $C(\theta, \eta, c_0, m, p_0)$ such that

$$\left(\iint_{U(\theta)} w^{p_0} d\mu dt \right)^{\frac{1}{p_0}} \leq C |U(1)|_{d\mu \times dt}^{\frac{1}{p_0}}.$$

Hereafter, we will make use of the following notation. Given $r > 0$, we define

$$(25) \quad I_-(r) := (-r^{2s}, 0), \quad I_+(r) := (0, r^{2s}),$$

$$(26) \quad Q_-(r) := B_r(0) \times I_-(r), \quad Q_+(r) := B_r(0) \times I_+(r).$$

The first step to prove Theorem 4.4 is to establish the next estimate (see [24, Proposition 3.4]). Notice that we just have to consider the case where $B_r(x_0) = B_r(0)$. For simplicity, we will write B_r instead of $B_r(0)$.

Lemma 4.6. *Assume that $\frac{1}{2} \leq r < R \leq 1$ and let $p > 0$. Consider $v \geq 0$ a supersolution to (22), then*

$$(27) \quad \left(\iint_{Q_-(r)} v^{-\tau p} d\mu dt \right)^{\frac{1}{\tau}} \leq A \iint_{Q_-(R)} v^{-p} d\mu dt,$$

where $\tau := 1 + \frac{2s}{N}$ and

$$A := A(N, s, p, r, R, \gamma) = C(N, s, \gamma)(p+1)^2 \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s} - r^{2s}} \right)^\tau.$$

Proof. Without loss of generality we can assume that $v \geq \varepsilon > 0$ in $Q_-(R)$ (otherwise we can deal with $v + \varepsilon$ and let $\varepsilon \rightarrow 0$ at the end). Let $q > 1$ and $\psi \in Y_0^{s, \gamma}(B_R) \cap L^\infty(B_R)$ be a nonnegative radial cutoff function such that $\text{supp}(\psi) \subseteq B_R$ with $r < R$, $0 \leq \psi \leq 1$, $\psi = 1$ in B_r and

$$(28) \quad \frac{(\psi(x) - \psi(y))^2}{|x - y|^2} \leq \frac{C}{(R - r)^2}.$$

Using $\psi^{q+1} v^{-q}$ as a test function in (22), it follows that

$$\int_{B_R} \psi^{q+1} v^{-q} v_t d\mu + \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v(x, t) - v(y, t)) (\psi^{q+1}(x) v^{-q}(x, t) - \psi^{q+1}(y) v^{-q}(y, t)) d\nu \geq 0.$$

Hence, using a pointwise inequality proved in [24, Lemma 3.3], it can be deduced that

$$\begin{aligned} & \frac{1}{q-1} \int_{B_R} \psi^{q+1} (v^{1-q})_t d\mu + \frac{1}{q-1} \frac{a_{N,s}}{2} \int_{B_r} \int_{B_r} (v^{\frac{1-q}{2}}(x, t) - v^{\frac{1-q}{2}}(y, t))^2 d\nu \\ & \leq Cq \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\left(\frac{v(x, t)}{\psi(x)} \right)^{1-q} + \left(\frac{v(y, t)}{\psi(y)} \right)^{1-q} \right) (\psi(x) - \psi(y))^2 d\nu. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\left(\frac{v(x, t)}{\psi(x)} \right)^{1-q} + \left(\frac{v(y, t)}{\psi(y)} \right)^{1-q} \right) (\psi(x) - \psi(y))^2 d\nu \\ & \leq 2 \int_{B_R} \int_{B_R} \frac{v^{1-q}(x, t) (\psi(x) - \psi(y))^2 dx dy}{|x|^\gamma |y|^\gamma |x - y|^{N+2s}} \\ & \quad + 4 \int_{\mathbb{R}^N \setminus B_R} \int_{B_R} \frac{v^{1-q}(x, t) (\psi(x) - \psi(y))^2 dx dy}{|x|^\gamma |y|^\gamma |x - y|^{N+2s}}. \end{aligned}$$

We set

$$I = \int_{B_R} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}},$$

and

$$J = \int_{\mathbb{R}^N \setminus B_R} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}}.$$

Let begin by estimating the term J . Taking into consideration that $\frac{1}{|y|^\gamma} \leq \frac{1}{|x|^\gamma}$ for $x \in B_R$ and $y \in \mathbb{R}^N \setminus B_R$ and using Fubini, we reach that

$$\begin{aligned} J &\leq \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{\mathbb{R}^N \setminus B_R\} \cap \{|x-y| > R-r\}} \frac{(\psi(x) - \psi(y))^2 dy dx}{|x - y|^{N+2s}} \\ &+ \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{\mathbb{R}^N \setminus B_R\} \cap \{|x-y| \leq R-r\}} \frac{(\psi(x) - \psi(y))^2 dy dx}{|x - y|^{N+2s}} \\ &\leq J_1 + J_2. \end{aligned}$$

Setting $\rho = |x - y|$, we get

$$\begin{aligned} J_1 &\leq 4 \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{\mathbb{R}^N \setminus B_R\} \cap \{|x-y| > R-r\}} \frac{dy dx}{|x - y|^{N+2s}} \\ &\leq 4 \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx \int_{R-r}^\infty \rho^{-1-2s} d\rho \\ &\leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx. \end{aligned}$$

We estimate now the term J_2 . Using (28), it follows that

$$\begin{aligned} J_2 &\leq \frac{C}{(R-r)^2} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{\mathbb{R}^N \setminus B_R\} \cap \{|x-y| \leq R-r\}} \frac{|x - y|^2 dy dx}{|x - y|^{N+2s}} \\ &\leq \frac{C}{(R-r)^2} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx \int_0^{R-r} \rho^{1-2s} d\rho \\ &\leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx, \end{aligned}$$

with C possibly changing from line to line. Hence

$$J \leq \frac{C}{(R-r)^{2s}} \int_{B_R} v^{1-q}(x, t) d\mu.$$

We deal now with I . Using the definition of ψ , we get easily that

$$\begin{aligned}
I &= \iint_{B_R \times B_R \setminus B_r \times B_r} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}} \\
&= \int_{B_R \setminus B_r} \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}} + \int_{B_R \setminus B_r} \int_{B_R \setminus B_r} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}} \\
&+ \int_{B_r} \int_{B_R \setminus B_r} \frac{v^{1-q}(x, t)}{|x|^\gamma} \frac{(\psi(x) - \psi(y))^2 dx dy}{|y|^\gamma |x - y|^{N+2s}} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let begin by estimate I_1 . Since $(x, y) \in B_r \times B_R \setminus B_r$, then $|x| \leq |y|$, hence

$$\begin{aligned}
I_1 &\leq \int_{B_R \setminus B_r} \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \frac{(\psi(x) - \psi(y))^2 dx dy}{|x - y|^{N+2s}} \\
&\leq \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{B_R \setminus B_r\} \cap \{|x-y| > R-r\}} \frac{(\psi(x) - \psi(y))^2 dy dx}{|x - y|^{N+2s}} \\
&+ \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{B_R \setminus B_r\} \cap \{|x-y| \leq R-r\}} \frac{(\psi(x) - \psi(y))^2 dy dx}{|x - y|^{N+2s}} \\
&\leq I_{11} + I_{12}.
\end{aligned}$$

As in the previous computations, by setting $\rho = |x - y|$ and using the fact that ψ is bounded, we conclude that

$$I_{11} \leq 4 \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{R-r}^{\infty} \rho^{-1-2s} d\rho \leq \frac{C}{(R-r)^{2s}} \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx.$$

In the same way and using the fact that $\frac{(\psi(x) - \psi(y))^2}{|x - y|^2} \leq \frac{C}{(R-r)^2}$, we get

$$I_{12} \leq \frac{C}{(R-r)^2} \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_0^{R-r} \rho^{1-2s} d\rho \leq \frac{C}{(R-r)^{2s}} \int_{B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx.$$

Therefore,

$$I_1 \leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx.$$

We deal now with I_2 . Since $\frac{1}{2} \leq r \leq |x|, |y| \leq R < 1$, then

$$\begin{aligned}
I_2 &\leq 2^\gamma \int_{B_R \setminus B_r} \int_{B_R \setminus B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \frac{(\psi(x) - \psi(y))^2 dx dy}{|x - y|^{N+2s}} \\
&\leq \int_{B_R \setminus B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{B_R \setminus B_r\} \cap \{|x-y| > R-r\}} \frac{(\psi(x) - \psi(y))^2 dy}{|x - y|^{N+2s}} \\
&\quad + \int_{B_R \setminus B_r} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \int_{\{B_R \setminus B_r\} \cap \{|x-y| \leq R-r\}} \frac{(\psi(x) - \psi(y))^2 dy}{|x - y|^{N+2s}} \\
&\leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx,
\end{aligned}$$

by repeating the computations performed for I_1 . Let consider now the term I_3 which is the most complicated.

$$\begin{aligned}
I_3 &= \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \left(\int_{|y| \leq \frac{|x|}{2}} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{N+2s} |y|^\gamma} dy \right) dx \\
&\quad + \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \left(\int_{\frac{|x|}{2} \leq |y| \leq r} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{N+2s} |y|^\gamma} dy \right) dx \\
&= I_{31} + I_{32}.
\end{aligned}$$

If $|y| \leq \frac{|x|}{2}$, then $|x - y| \geq \frac{|x|}{2} \geq \frac{r}{2} \geq \frac{1}{4}$, and thus,

$$\begin{aligned}
I_{31} &\leq C \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \left(\int_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^\gamma} dy \right) dx \\
&\leq C \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^\gamma} \left(\int_0^{\frac{|x|}{2}} \rho^{N-1-\gamma} d\rho \right) dx \\
&\leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx.
\end{aligned}$$

To estimate I_{32} , we use the fact that $\frac{1}{|y|^\gamma} \leq \frac{2^\gamma}{|x|^\gamma}$, hence

$$\begin{aligned}
I_{32} &\leq C \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \left(\int_{\frac{|x|}{2} \leq |y| \leq r} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{N+2s}} dy \right) dx \\
&\leq C \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \left(\int_{\{\frac{|x|}{2} \leq |y| \leq r\} \cap \{|x-y| > R-r\}} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{N+2s}} dy \right) dx \\
&\quad + C \int_{r \leq |x| \leq R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} \left(\int_{\{\frac{|x|}{2} \leq |y| \leq r\} \cap \{|x-y| \leq R-r\}} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{N+2s}} dy \right) dx.
\end{aligned}$$

Using the same computations as in the estimates of I_{11} and I_{12} it follows that

$$I_{32} \leq \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x, t)}{|x|^{2\gamma}} dx.$$

Combining the estimates above, there results that

$$\begin{aligned} \int_{B_R} \psi^{q+1} (v^{1-q})_t d\mu + \frac{a_{N,s}}{2} \int_{B_r} \int_{B_r} (v^{\frac{1-q}{2}}(x,t) - v^{\frac{1-q}{2}}(y,t))^2 d\nu \\ \leq \frac{Cq^2}{(R-r)^{2s}} \frac{a_{N,s}}{2} \int_{B_R} v^{1-q}(x,t) d\mu. \end{aligned}$$

Set now $\theta(t) := [\min\{\frac{t+R^{2s}}{R^{2s}-r^{2s}}, 1\}]_+$. Then multiplying the last inequality by θ , integrating in time in $(-R^{2s}, t)$ with $t \in (-r^{2s}, 0)$, and noticing that $\theta(t) = 1$ for $t \geq -r^{2s}$ and $|\theta'(t)| \leq \frac{1}{R^{2s}-r^{2s}}$, it follows that

$$\begin{aligned} (29) \quad \sup_{t \in I_-(r)} \int_{B_r} (v^{1-q}) d\mu + \frac{a_{N,s}}{2} \iint_{Q_-(r)} \int_{B_r} (v^{\frac{1-q}{2}}(x,t) - v^{\frac{1-q}{2}}(y,t))^2 d\nu dt \\ \leq Cq^2 \frac{a_{N,s}}{2} \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s}-r^{2s}} \right) \iint_{Q_-(R)} v^{1-q}(x,t) d\mu dt. \end{aligned}$$

Recalling that $\tau := 1 + \frac{2s}{N}$, let us define $w := v^{\frac{1-q}{2}}$. Then, applying Hölder inequality, we get

$$\begin{aligned} \iint_{Q_-(r)} w^{2\tau} d\mu dt &= \iint_{Q_-(r)} w^2 w^{\frac{4s}{N}} d\mu dt \\ &\leq \int_{I_-(r)} \left(\int_{B_r} w^2 d\mu \right)^{\frac{2s}{N}} \left(\int_{B_r} w^{2^*} d\mu \right)^{\frac{2}{2^*}} dt. \end{aligned}$$

Since $\gamma > 0$ and $R \leq 1$, we conclude that

$$\iint_{Q_-(r)} w^{2\tau} d\mu dt \leq \int_{I_-(r)} \left(\int_{B_r} w^2(x,t) d\mu \right)^{\frac{2s}{N}} \left(\int_{B_r} \frac{w^{2^*}}{|x|^{2s}\gamma} dx \right)^{\frac{2}{2^*}} dt.$$

Now, using the Sobolev inequality obtained in Theorem B.9 in the Appendix,

$$\begin{aligned} \iint_{Q_-(r)} w^{2\tau} d\mu dt &\leq C \sup_{t \in I_-(r)} \left(\int_{B_r} w^2(x,t) d\mu \right)^{\frac{2s}{N}} \\ &\quad \times \left(\iint_{Q_-(r)} \int_{B_r} (w(x,t) - w(y,t))^2 d\nu dt + r^{-2s} \iint_{Q_-(r)} w^2 d\mu dt \right). \end{aligned}$$

Applying (29) twice at this inequality, and recalling that $\frac{1}{2} \leq r \leq 1$, it can be checked that

$$\iint_{Q_-(r)} w^{2\tau} d\mu dt \leq A(q, r, R, N, s) \left(\iint_{Q_-(R)} w^2 d\mu dt \right)^\tau$$

where

$$A(q, r, R, N, s) = C(N, s, \gamma) q^2 \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s}-r^{2s}} \right)^\tau.$$

Setting $p := q - 1$, we conclude the proof. \square

As an application of the previous estimate, we reach a control of $\sup_{Q_-(r)} v^{-1}$. More precisely, we have the following result.

Lemma 4.7. *Assume that $\frac{1}{2} \leq r < R \leq 1$ and $p \in (0, 1]$. Then, there exists a constant $C = C(N, s, \gamma) > 0$ such that every $v \geq 0$ supersolution to the problem (22) satisfies*

$$(30) \quad \sup_{Q_-(r)} v^{-1} \leq \left(\frac{C}{\alpha(r, R)} \right)^{\frac{1}{p}} \left(\iint_{Q_-(R)} v^{-p} d\mu dt \right)^{\frac{1}{p}},$$

where

$$\alpha(r, R) = \begin{cases} (R-r)^{N+2s} & \text{if } s \geq \frac{1}{2}, \\ (R^{2s} - r^{2s})^{\frac{N+2s}{2s}} & \text{if } s < \frac{1}{2}. \end{cases}$$

Proof. We consider

$$\mathcal{M}(r, p) := \left(\iint_{Q_-(r)} v^{-p} d\mu dt \right)^{\frac{1}{p}}.$$

By Lemma 4.6, we have

$$\mathcal{M}(r, \tau p) \leq A^{\frac{1}{p}} \mathcal{M}(R, p),$$

where $\tau := 1 + \frac{2s}{N}$. Construct now the sequences $\{r_i\}_{i \in \mathbb{N}}$ and $\{p_i\}_{i \in \mathbb{N}}$ by setting $r_0 := R > r_1 > r_2 > \dots > r$ and $p_i := p\tau^i$. Using again Lemma 4.6, we obtain

$$\mathcal{M}(r, p_{m+1}) \leq \mathcal{M}(r_{m+1}, p_{m+1}) \leq A_m^{\frac{1}{\tau^{m+1}p}} \mathcal{M}(r_m, p_m),$$

with $A_m := C(p_m + 1)^2 ((r_m - r_{m+1})^{-2s} + (r_m^{2s} - r_{m+1}^{2s})^{-1})$. Iterating this and following the arguments in [24, Theorem 3.5], we conclude the result. \square

We prove now a control of a small positive exponents of u .

Lemma 4.8. *Suppose that $\frac{1}{2} \leq r < R \leq 1$, and fix $q \in (0, \tau^{-1}]$, with $\tau := 1 + \frac{2s}{N}$. Then, if $v \geq 0$ is a supersolution to (22), we have*

$$(31) \quad \left(\iint_{Q_+(r)} v^{q\tau} d\mu dt \right)^{\frac{1}{\tau}} \leq \alpha \iint_{Q_+(R)} v^q d\mu dt,$$

where

$$\alpha = \alpha(N, s, r, R, \gamma) = C(N, s, \gamma) \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s} - r^{2s}} \right).$$

Proof. The proof follows similarly to the one of Lemma 4.6 (see [24, Proposition 3.6] for a detailed proof with the Lebesgue measure). We set $a := (1 - q) \in [1 - \tau^{-1}, 1)$ and $w(x, t) := v^{\frac{1-a}{2}}$. Then, using $v^{-a}\psi^2$ (with ψ defined in the proof of Lemma 4.6) as a test function in (22), we reach that

$$\begin{aligned} & \sup_{t \in I_+(r)} \int_{B_r} w^2(x, t) d\mu + C(q) \frac{a_{N,s}}{2} \int_{Q_+(r)} \int_{B_r} (w^2(x, t) - w^2(y, t))^2 dv dt \\ & \leq C(q) \frac{a_{N,s}}{2} \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s} - r^{2s}} \right) \int_{Q_+(R)} w^2(x, t) d\mu dt. \end{aligned}$$

Using Theorem B.9 and proceeding as in Lemma 4.6, we get (31). \square

Define

$$\mathcal{H}(r, q) = \left(\int_{Q_+(r)} v^q d\mu dt \right)^{\frac{1}{q}}.$$

From (31), we get $\mathcal{H}(r, \tau q) \leq \alpha^{\frac{1}{q}} \mathcal{H}(R, q)$. Let define $q_j := \tau^{-j}$ and

$$r_j := \begin{cases} r + \frac{R-r}{2^j} & \text{if } s \geq \frac{1}{2}, \\ \left(r^{2s} + \frac{R^{2s} - r^{2s}}{2^j} \right)^{1/2s} & \text{if } s < \frac{1}{2}. \end{cases}$$

By Lemma 4.8 for r_n and r_{n-1} , it follows that

$$(32) \quad \mathcal{H}(r_n, q_1 \tau) \leq \alpha_n^\tau \mathcal{H}(r_{n-1}, q_1),$$

where $\alpha_n = C(N, s, \gamma) \left(\frac{1}{(r_{n-1} - r_n)^{2s}} + \frac{1}{r_{n-1}^{2s} - r_n^{2s}} \right)$. By using the definition of r_n and considering that $r \geq \frac{1}{2}$, we get

$$\alpha_n \leq C(N, s, \gamma) \frac{2^{2ns}}{(R - r)^{2s}}.$$

Hence

$$(33) \quad \mathcal{H}(r_n, q_1 \tau) \leq \left(C(N, s, \gamma) \frac{2^{2ns}}{(R - r)^{2s}} \right)^\tau \mathcal{H}(r_{n-1}, q_1).$$

Iterating this inequality (see [24, Theorem 3.7]), we reach the next result.

Lemma 4.9. *Assume that $\frac{1}{2} \leq r < R \leq 1$ and $q \in (0, \tau^{-1})$, with $\tau := 1 + \frac{2s}{N}$. Then, every supersolution $v \geq 0$ of problem (22) satisfies*

$$(34) \quad \iint_{Q_+(r)} v \, d\mu dt \leq \left(\frac{C}{|Q_+(1)|_{d\mu \times dt} \overline{\alpha}(r, R)} \right)^{\frac{1-q}{q}} \left(\iint_{Q_+(R)} v^q \, d\mu dt \right)^{\frac{1}{q}},$$

where $C = C(N, s, \gamma) > 0$ and

$$\overline{\alpha}(r, R) = \begin{cases} (R - r)^{\omega_1} & \text{if } s \geq \frac{1}{2}, \\ (R^{2s} - r^{2s})^{\omega_2} & \text{if } s < \frac{1}{2}, \end{cases}$$

with $\omega_1, \omega_2 > 0$ depending only on s, N .

In order to apply Lemma 4.5 we need to estimate

$$(35) \quad |Q_+(1) \cap \{\log v < -m - a\}|_{d\mu \times dt} \quad \text{and} \quad |Q_-(1) \cap \{\log v > m - a\}|_{d\mu \times dt},$$

where $m, a > 0$ and v is a supersolution to (22). To do this, we need the following auxiliary result, whose proof immediately follows from Lemma 4.1 in [24].

Lemma 4.10. *Let $I \subset \mathbb{R}$ and $\psi : \mathbb{R}^N \rightarrow [0, +\infty)$ be a continuous function satisfying $\text{supp}(\psi) = \overline{B_R}$ for some $R > 0$ and $\|\psi\|_{Y_0^{s, \gamma}(\mathbb{R}^N)} \leq C$. Then, for $v : \mathbb{R}^N \times I \rightarrow [0, +\infty)$, the following inequality holds,*

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v(x, t) - v(y, t)) (-\psi^2(x) v^{-1}(x, t) + \psi^2(y) v^{-1}(y, t)) \, d\nu \\ & \geq \int_{B_R} \int_{B_R} \psi(x) \psi(y) \left(\log \frac{v(y, t)}{\psi(y)} - \log \frac{v(x, t)}{\psi(x)} \right)^2 \, d\nu - 3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\psi(x) - \psi(y))^2 \, d\nu. \end{aligned}$$

With this result, we can establish the estimates in (35) (see [24, Proposition 4.2] for more details).

Lemma 4.11. *Assume that v is a supersolution to (22) in the cylinder $Q := B_2 \times (-1, 1)$, then there exists a positive constant $C = C(N, s, \gamma)$ such that for some constant $a = a(v)$, we have*

$$(36) \quad \forall m > 0 : |Q_+(1) \cap \{\log v < -m - a\}|_{d\mu \times dt} \leq \frac{C|B_1|_{d\mu}}{m},$$

and

$$(37) \quad \forall m > 0 : |Q_-(1) \cap \{\log v > m - a\}|_{d\mu \times dt} \leq \frac{C|B_1|_{d\mu}}{m}.$$

Proof. Suppose that $v \geq \varepsilon > 0$ in Q . Let ψ be such that $\psi^2 = ((\frac{3}{2} - \frac{|x|}{2}) \wedge 1) \vee 0$, and let us denote $w(x, t) := -\log \frac{v(x, t)}{\psi(x)}$. Using $\frac{\psi^2}{v}$ as a test function in (22) and noticing that $\text{supp}(\psi^2) \subseteq B_{3/2}$, by Lemma 4.10 and the fact that $\|\psi\|_{Y_0^{s, \gamma}(\mathbb{R}^N)} \leq C$, there follows

$$\begin{aligned} \int_{B_{3/2}} \psi^2 w_t d\mu + \frac{a_{N,s}}{2} \int_{B_{3/2}} \int_{B_{3/2}} \psi(x) \psi(y) (w(x, t) - w(y, t))^2 d\nu \\ \leq 3 \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\psi(x) - \psi(y))^2 d\nu \leq C. \end{aligned}$$

We set $W(t) := \frac{\int_{B_{3/2}} \psi^2 w(x, t) d\mu}{\int_{B_{3/2}} \psi^2 d\mu}$, then by the Poincaré type inequality obtained in Theorem B.10, we reach that

$$\int_{B_{3/2}} \psi^2 w_t d\mu + C \int_{B_{3/2}} (w(x, t) - W(t))^2 \psi(x)^2 d\mu \leq C.$$

Let $(t_1, t_2) \subset (-1, 1)$. Integrating in time the previous inequality, dividing by $\int_{B_{3/2}} \psi^2 d\mu$, and noticing that

$$\int_{B_{3/2}} \psi^2 d\mu \leq 2^{N-2\gamma} |B_1|_{d\mu},$$

one gets

$$\frac{W(t_2) - W(t_1)}{t_2 - t_1} + \frac{C_1}{|B_1|_{d\mu}(t_2 - t_1)} \int_{t_1}^{t_2} \int_{B_1} (w(x, t) - W(t))^2 d\mu \leq C_2.$$

We can suppose that W is differentiable. In the contrary case, it is possible to follow a discretization argument as in [24, Proposition 4.2]. By letting $t_2 \rightarrow t_1$, we get

$$(38) \quad W'(t) + \frac{C_1}{|B_1|_{d\mu}} \int_{B_1} (w(x, t) - W(t))^2 d\mu \leq C_2 \quad a.e \text{ in } (-1, 1).$$

Define $\tilde{W}(t) = W(t) - C_2 t$ and $\tilde{w}(x, t) = w(x, t) - C_2 t$. From (38), it follows that

$$(39) \quad \tilde{W}'(t) + \frac{C_1}{|B_1|_{d\mu}} \int_{B_1} (\tilde{w}(x, t) - \tilde{W}(t))^2 d\mu \leq 0 \quad a.e \text{ in } (-1, 1).$$

Notice that from (39) we deduce $\tilde{W}'(t) \leq 0$, and therefore calling $a(v) := W(0)$ there results

$$\tilde{W}(t) \leq W(0) =: a(v) \text{ for all } t \in (0, 1).$$

Let $t \in (0, 1)$. Then if we define

$$G_m^+(t) := \{x \in B_1(0) : \tilde{w}(x, t) > m + a\},$$

for $x \in G_m^+(t)$, we have

$$\tilde{w}(x, t) - \tilde{W}(t) \geq m + a - \tilde{W}(t) > 0.$$

Thus

$$\tilde{W}'(t) + \frac{C_1}{|B_1|_{d\mu}} |G_m^+(t)|_{d\mu} (m + a - \tilde{W}(t))^2 \leq 0.$$

Hence

$$\frac{-\tilde{W}'(t)}{(m + a - \tilde{W}(t))^2} \geq \frac{C_1}{|B_1|_{d\mu}} |G_m^+(t)|_{d\mu}.$$

Integrating the previous differential inequality for $t \in (0, 1)$ and substituting \tilde{w} by its value, yields

$$|Q_+(1) \cap \{\log v + C_2 t < -m - a\}|_{d\mu \times dt} \leq \frac{C_1 |B_1|_{d\mu}}{m}.$$

Now

$$\begin{aligned}
|Q_+(1) \cap \{\log v < -m - a\}|_{d\mu \times dt} &\leq |Q_+(1) \cap \{\log v + C_2 t < -\frac{m}{2} - a\}|_{d\mu \times dt} \\
&+ |Q_+(1) \cap \{C_2 t > \frac{m}{2}\}|_{d\mu \times dt} \\
&\leq \frac{C|B_1|_{d\mu}}{m},
\end{aligned}$$

what finishes the proof of (36). Estimate (37) follows using the same approach. \square

We are now able to prove the weighted weak Harnack inequality.

Proof of Theorem 4.4. Roughly speaking, the key to prove this result will be to define appropriate functions and parameters so that we can deduce the result from Lemma 4.5. Indeed, we divide the proof in two cases. Let $0 < r < 1$ such that $B_r \subset \Omega$.

- (1) Assume first that $s \geq \frac{1}{2}$.

We set $\theta_1 = \theta_2 = \frac{1}{2}$ and define $U_1(r) = B_r \times (1 - r^{2s}, 1)$, $U_2(r) = B_r \times (-1, -1 + r^{2s})$. In the same way we consider $U_1(1) = Q_+(1)$ and $U_2(1) = Q_-(1)$.

Let $w_1 := e^{-a}v^{-1}$, $w_2 := e^a v$ where $a = a(v)$ was defined in Lemma 4.11. From this result we obtain that

$$|Q_+(1) \cap \{\log w_1 > m\}|_{d\mu \times dt} \leq \frac{C|B_1|_{d\mu}}{m},$$

and

$$|Q_-(1) \cap \{\log w_2 > m\}|_{d\mu \times dt} \leq \frac{C|B_1|_{d\mu}}{m}.$$

Using Lemma 4.7, it follows that $(w_1, U_1(r))$ satisfies the conditions of Lemma 4.5 with $p_0 = \infty$ and η any positive constant. Moreover, by Lemma 4.9, $(w_2, U_2(r))$ satisfies the same conditions with $p_0 = 1$ and $\eta = \frac{N}{N+2s} < 1$. Hence we conclude that

$$\sup_{U_1(\frac{1}{2})} w_1 \leq C \quad \text{and} \quad \|w_2\|_{L^1(U_2(\frac{1}{2}), d\mu)} \leq \tilde{C}.$$

Using these estimates and the definitions of w_1 and w_2 , we get

$$\|v\|_{L^1(U_2(\frac{1}{2}), d\mu)} \leq C \inf_{U_1(\frac{1}{2})} v,$$

and the result follows in this case.

- (2) If $0 < s < \frac{1}{2}$, we have to change the domains by setting $\theta_1 = \theta_2 = (\frac{1}{2})^{2s}$ and $U_1(r) = B_{r^{\frac{1}{2s}}} \times (1 - r, 1)$, $U_2(r) = B_{r^{\frac{1}{2s}}} \times (-1, -1 + r)$. Then the same arguments as in the previous case allow us to conclude. \square

From the weighted weak Harnack inequality, we immediately deduce the next Corollary.

Corollary 4.12. *Let $\lambda \leq \Lambda_{N,s}$. Assume that u is a nonnegative function such that $u \not\equiv 0$, $u \in L^1(\Omega \times (0, T))$ and $\frac{u}{|x|^{2s}} \in L^1(\Omega \times (0, T))$. If u satisfies $u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \geq 0$ in the weak sense in $\Omega \times (0, T)$, then there exists $r_1 > 0$ and $t_2 > t_1 > 0$, and a constant $C = C(N, r_1, t_1, t_2)$ such that for each cylinder $B_r(0) \times (t_1, t_2) \subset \subset \Omega \times (0, T)$, $0 < r < r_1$,*

$$u \geq C|x|^{-\frac{N-2s}{2}+\alpha} \text{ in } B_r(0) \times (t_1, t_2),$$

where α is the singularity of the homogeneous problem given in Lemma 3.1. In particular, for r conveniently small we can assume that $u > 1$ in $B_r(0) \times (t_1, t_2)$.

Finally, to end this section, we can establish a boundedness condition on the solutions of (22).

Proposition 4.13. *Let $v \in \mathcal{C}([0, T]; L^2(\mathbb{R}^N, d\mu)) \cap L^2(0, T; Y_0^{s, \gamma}(\mathbb{R}^N))$ be a solution to (22) with $u_0 \in L^\infty(\Omega)$. If $f \in L^r(0, T; L^q(\Omega))$ with $r, q > 1$ and $\frac{1}{r} + \frac{N}{2qs} < 1$, then $v \in L^\infty(\Omega \times (0, T))$.*

Proof. The proof follows the same idea of the classical result by D.G. Aronson and J.Serrin in [5]. We test with $G_k(v) \in Y_0^{s, \gamma}(\Omega)$ in (22), and defining

$$||G_k(v)||^2 := \|G_k(v)\|_{L^\infty(0, T; L^2(\Omega, d\mu))}^2 + \|G_k(v)\|_{L^2(0, T; Y_0^{s, \gamma}(\Omega))}^2,$$

the result is obtained as a simplified version of [34, Theorem 29]. The presence of the singular term is handled in a straightforward way, so we skip the details. \square

Remark 4.14. Apart from the integral version we proved, if the solution is bounded we can prove the strong Harnack inequality by classical arguments. We skip the details because we do not use this inequality in the applications.

Corollary 4.15. *If u is a solution to (1) with a sufficient regular datum f , then $u \leq C|x|^{-\frac{N-2s}{2}+\alpha}$ in $\Omega \times (0, T)$.*

5. THE LINEAR PROBLEM: DEPENDENCE ON THE SPECTRAL PARAMETER λ .

Along this section we will study the problem

$$(40) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + g(x, t) \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \geq 0 \text{ if } x \in \Omega, \end{cases}$$

where $g(x, t)$ is a nonnegative function. The goal will be to establish some necessary and sufficient conditions on g and u_0 in order to find solutions of this problem. These results correspond to the ones obtained by P. Baras and J. A. Goldstein for the heat equation in presence of the inverse square potential (see [9]).

First, we deal with the necessary summability conditions on g and u_0 .

Theorem 5.1. *Let $0 < \lambda \leq \Lambda_{N, s}$. Assume that \tilde{u} is a positive weak supersolution to the problem (40). Then g and u_0 must satisfy*

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\gamma} g \, dx \, dt < +\infty, \quad \int_{B_r(0)} |x|^{-\gamma} u_0 \, dx < +\infty,$$

for any cylinder $B_r(0) \times (t_1, t_2) \subset \subset \Omega \times (0, T)$, where γ was defined in (18).

Proof. Fix $\varepsilon > 0$. Let consider φ_n , the positive solution to

$$(41) \quad \begin{cases} -(\varphi_n)_t + (-\Delta)^s \varphi_n &= \lambda \frac{\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} + 1 \text{ in } \Omega \times (-\varepsilon, T), \\ \varphi_n &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi_n(x, T) &= 0 \text{ in } \Omega, \end{cases}$$

with

$$(42) \quad \begin{cases} -(\varphi_0)_t + (-\Delta)^s \varphi_0 &= 1 \text{ in } \Omega \times (-\varepsilon, T), \\ \varphi_0 &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi_0(x, T) &= 0 \text{ in } \Omega. \end{cases}$$

Notice that φ_0 is a strong solution in $\Omega \times (-\varepsilon, T]$, and therefore every φ_n is a strong solution too (see Appendix A). Furthermore, $\varphi_{n-1} \leq \varphi_n \leq \varphi$, where φ is the positive solution to

$$\begin{cases} -\varphi_t + (-\Delta)^s \varphi &= \lambda \frac{\varphi}{|x|^{2s}} = 1 \text{ in } \Omega \times (-\varepsilon, T), \\ \varphi &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi(x, T) &= 0, \quad \Omega. \end{cases}$$

As a consequence of Corollary 4.12, for any cylinder $C_{r_1, t_1, t_2} := B_r(0) \times (t_1, t_2) \subset \subset \Omega \times (-\varepsilon, T)$, $0 < r < r_1$, there exists a constant $A = A(N, s, C_{r_1, t_1, t_2})$, such that

$$(43) \quad \varphi(x, t) \geq \frac{A}{|x|^\gamma}, \quad (x, t) \in C_{r_1, t_1, t_2}, \quad \gamma = \frac{N-2s}{2} - \alpha.$$

Since φ_n is regular and bounded we can use it as a test function in (40), hence we get

$$\begin{aligned} & \int_0^T \int_\Omega g \varphi_n dx dt + \int_\Omega u_0 \varphi_n(x, 0) dx \\ & \leq - \int_0^T \int_\Omega \tilde{u}(\varphi_n)_t dx dt + \int_0^T \int_{\mathbb{R}^N} \tilde{u}(-\Delta)^s \varphi_n dx dt - \lambda \int_0^T \int_\Omega \frac{\tilde{u} \varphi_n}{|x|^{2s}} dx dt \\ & \leq - \int_0^T \int_\Omega \tilde{u}(\varphi_n)_t dx dt + \int_0^T \int_\Omega \tilde{u}(-\Delta)^s \varphi_n dx dt - \lambda \int_0^T \int_\Omega \frac{\tilde{u} \varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} dx dt \\ & = \int_0^T \int_\Omega \tilde{u} dx dt = C < +\infty. \end{aligned}$$

Since both integrals in the left hand side are positive, in particular each one is uniformly bounded. Thus, $\{g \varphi_n\}$ is an increasing sequence uniformly bounded in $L^1(\Omega \times (0, T))$, and applying the Monotone Convergence Theorem and (43) we get

$$\begin{aligned} C \int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\frac{N-2s}{2} + \alpha} g dx dt & \leq \int_{t_1}^{t_2} \int_{B_r(0)} g \varphi dx dt \\ & \leq \int_0^T \int_\Omega g \varphi dx dt = \lim_{n \rightarrow \infty} \int_0^T \int_\Omega g \varphi_n dx dt < +\infty. \end{aligned}$$

Likewise, $\{u_0 \varphi_n(x, 0)\}$ is also an increasing sequence, uniformly bounded in $L^1(\Omega)$, and thus, choosing t_1 and t_2 so that $0 \in (t_1, t_2) \subset (-\varepsilon, T)$, as above we conclude

$$\tilde{C} \int_{B_r(0)} |x|^{-\frac{N-2s}{2} + \alpha} u_0(x) dx \leq \int_\Omega u_0 \varphi(x, 0) dx < +\infty.$$

□

Conversely, we would like to find the optimal summability conditions on g and u_0 to prove existence of weak solution. In this direction, notice that if $g \in L^2(0, T; H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$, by Remark 2.8 we can assure the existence of an energy solution of the problem (40) whether $\lambda < \Lambda_{N,s}$, and in $H(\Omega)$ (see (10)) for $\lambda = \Lambda_{N,s}$. A sharper result, for a more general class of data, is the following.

Theorem 5.2. *Assume $0 < \lambda \leq \Lambda_{N,s}$, and that g and u_0 satisfy*

$$\int_\Omega \frac{u_0}{|x|^\gamma} dx < +\infty, \quad \int_0^T \int_\Omega \frac{g}{|x|^\gamma} dx dt < +\infty,$$

where γ was defined in (18). Then problem (40) has a positive weak solution.

Proof. Consider the approximated problems

$$(44) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n &= \lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + g_n \text{ in } \Omega \times (0, T), \\ u_n(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u_n(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u_n(x, 0) &= T_n(u_0(x)) \text{ if } x \in \Omega, \end{cases}$$

where

$$(45) \quad \begin{cases} u_{0t} + (-\Delta)^s u_0 &= g_1 \text{ in } \Omega \times (0, T), \\ u_0(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u_0(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u_0(x, 0) &= T_1(u_0(x)) \text{ if } x \in \Omega, \end{cases}$$

with $g_n = T_n(g)$ and

$$T_n(g) = \begin{cases} g & \text{if } |g| \leq n, \\ n \frac{g}{|g|} & \text{if } |g| > n. \end{cases}$$

By Lemma 2.9, it follows that $u_0 \leq u_1 \leq \dots \leq u_{n-1} \leq u_n$ in $\mathbb{R}^N \times (0, T)$. Note that, since the right hand sides of these problems are bounded, every u_n is actually an energy solution.

Consider φ the solution of the problem

$$(46) \quad \begin{cases} -\varphi_t + (-\Delta)^s \varphi - \lambda \frac{\varphi}{|x|^{2s}} &= 1 & \text{in } \Omega \times (-\varepsilon, T), \\ \varphi &> 0 & \text{in } \Omega \times (-\varepsilon, T), \\ \varphi &= 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi(x, T) &= C & \text{in } \Omega, \end{cases}$$

where $C > 0$. As a consequence of the weak Harnack inequality, Theorem 4.4, and Proposition 4.13, for any cylinder $B_r(0) \times [t_1, t_2] \subset \Omega \times (-\varepsilon, T)$ we find $c_1, c_2 > 0$ such that

$$(47) \quad \frac{c_1}{|x|^\gamma} \leq \varphi(x, t) \leq \frac{c_2}{|x|^\gamma}.$$

Since φ also belongs to $L^2(0, T; H_0^s(\Omega))$, we can use it as a test function in (44). Thus,

$$\begin{aligned} \int_0^T \int_\Omega (u_n)_t \varphi \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} u_n (-\Delta)^s \varphi \, dx \, dt &= \lambda \int_0^T \int_\Omega \frac{u_{n-1} \varphi}{|x|^{2s} + \frac{1}{n}} \, dx \, dt + \int_0^T \int_\Omega g_n \varphi \, dx \, dt \\ &\leq \lambda \int_0^T \int_\Omega \frac{u_n \varphi}{|x|^{2s}} \, dx \, dt + \int_0^T \int_\Omega g_n \varphi \, dx \, dt. \end{aligned}$$

Integrating in time and applying (46) and (47), we conclude that

$$\begin{aligned} C \int_\Omega u_n(x, T) \, dx + \int_0^T \int_\Omega u_n \, dx \, dt &\leq \int_0^T \int_\Omega g_n \varphi \, dx \, dt + \int_\Omega T_n(u_0(x)) \varphi(x, 0) \, dx \\ &\leq \int_0^T \int_\Omega g \varphi \, dx \, dt + \int_\Omega u_0(x) \varphi(x, 0) \, dx \\ &\leq C \int_0^T \int_\Omega \frac{g}{|x|^\gamma} \, dx \, dt + C \int_\Omega \frac{u_0(x)}{|x|^\gamma} \, dx \\ &< +\infty, \end{aligned}$$

by hypotheses. Hence, since the sequence $\{u_n\}_{n \in \mathbb{N}}$ is increasing, we can define $u := \lim_{n \rightarrow \infty} u_n$, and conclude that $u \in L^1(\Omega \times (0, T))$ by applying the Monotone Convergence Theorem.

Notice that, using the same computations as above and integrating in $\Omega \times [0, t]$ with $t \leq T$, by considering the estimates on $\{u_n\}_{n \in \mathbb{N}}$, we reach that

$$(48) \quad \sup_{t \in [0, T]} \int_\Omega u_n(x, t) \, dx + \int_0^T \int_\Omega u_n \, dx \, dt \leq C \text{ for all } n.$$

Fix $T_1 > T$, and define $\tilde{\varphi}$ as the unique solution to the problem

$$(49) \quad \begin{cases} -\tilde{\varphi}_t + (-\Delta)^s \tilde{\varphi} &= 1 & \text{in } \Omega \times (-\varepsilon, T_1), \\ \tilde{\varphi} &> 0 & \text{in } \Omega \times (-\varepsilon, T_1), \\ \tilde{\varphi} &= 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T_1], \\ \varphi(x, T_1) &= 0 & \text{in } \Omega. \end{cases}$$

It is clear that $\tilde{\varphi} \in L^\infty(\Omega \times (-\varepsilon, T_1))$ and $\tilde{\varphi}(x, t) \geq \bar{C} > 0$ for all $(x, t) \in B_r(0) \times [0, T]$, where $B_r(0) \subset \subset \Omega$. Now, using $\tilde{\varphi}$ as a test function in (44) and integrating in $\Omega \times (0, T)$, it follows that

$$\int_\Omega u_n(x, T) \tilde{\varphi}(x, T) dx dt + \int_0^T \int_\Omega u_n dx dt \geq \lambda \int_0^T \int_\Omega \frac{u_{n-1} \tilde{\varphi}}{|x|^{2s} + \frac{1}{n}} dx dt.$$

Thus

$$\lambda \int_0^T \int_\Omega \frac{u_{n-1} \tilde{\varphi}}{|x|^{2s} + \frac{1}{n}} dx dt \leq C \sup_{\{t \in [0, T]\}} \int_\Omega u_n(x, t) dx + \int_0^T \int_\Omega u_n dx dt \leq C \quad \text{for all } n.$$

Hence

$$\begin{aligned} \int_0^T \int_\Omega \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx dt &= \int_0^T \int_{B_r(0)} \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx dt + \int_0^T \int_{\Omega \setminus B_r(0)} \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx dt \\ &\leq C \int_0^T \int_\Omega \frac{u_{n-1} \tilde{\varphi}}{|x|^{2s} + \frac{1}{n}} dx dt + C \int_0^T \int_\Omega u_{n-1} dx dt \leq C. \end{aligned}$$

Therefore, by the Monotone Convergence Theorem we conclude that

$$\frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + g_n \uparrow \frac{u}{|x|^{2s}} + g \text{ strongly in } L^1(\Omega \times (0, T)).$$

To conclude that u is a weak solution to problem (40), it remains to check that $u \in \mathcal{C}([0, T]; L^1(\Omega))$. We claim that $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; L^1(\Omega))$, and hence the result follows. In order to prove this, we closely follow the arguments in [39].

For $n, m \in \mathbb{N}$, such that $n \geq m$, denote $u_{n,m} := u_n - u_m$, and $g_{n,m} := g_n - g_m$. Clearly, $u_{n,m}, g_{n,m} \geq 0$. We set

$$C_{n,m} := \lambda \int_0^t \int_\Omega \frac{u_{n,m}}{|x|^{2s}} dx d\tau + \int_0^t \int_\Omega g_{n,m} T_1(u_{n,m}) dx d\tau,$$

and then $C_{n,m} \rightarrow 0$ as $n, m \rightarrow \infty$.

By the definition of the approximated problems in (44) and the linearity of the operator, for $t \leq T$,

$$\int_0^t \int_\Omega (u_{n,m})_t T_1(u_{n,m}) dx d\tau + \int_0^t \int_\Omega (-\Delta)^s (u_{n,m}) T_1(u_{n,m}) dx d\tau \leq C_{n,m}.$$

Since $u_{n,m} \in L^2(0, T; H_0^s(\Omega))$, it follows that (see [34] for a detailed proof)

$$\int_0^t \int_\Omega (-\Delta)^s (u_{n,m}) T_1(u_{n,m}) dx d\tau \geq \int_0^t \|T_1(u_{n,m})\|_{H_0^s(\Omega)}^2 d\tau \geq 0,$$

and therefore,

$$\int_0^t \int_\Omega (u_{n,m})_t T_1(u_{n,m}) dx d\tau \leq C_{n,m}.$$

Let us define

$$\Psi(s) := \int_0^s T_1(\sigma) d\sigma.$$

Since $u_n \in \mathcal{C}([0, T]; L^2(\Omega))$, then

$$\int_0^t \int_{\Omega} (u_{n,m})_t T_1(u_{n,m}) dx d\tau = \int_{\Omega} (\Psi(u_{n,m})(t) - \Psi(u_{n,m})(0)) dx.$$

Thus

$$\int_{\Omega} \Psi(u_{n,m})(t) dx \leq C_{n,m} + \int_{\Omega} \Psi(u_{n,m})(0) dx.$$

Taking into account that $\Psi(u_{n,m})(0) = \Psi(T_n(u_0) - T_m(u_0))$ and by noticing that $\Psi(s) \leq |s|$ and $T_n(u_0) - T_m(u_0) \rightarrow 0$ strongly in $L^1(\Omega)$ as $n, m \rightarrow \infty$, we obtain that

$$\int_{\Omega} \Psi(u_{n,m})(t) dx \rightarrow 0 \text{ as } n, m \rightarrow \infty, \quad \text{uniformly in } t.$$

Now, since

$$\int_{|u_{n,m}| < 1} \frac{|u_{n,m}|^2(t)}{2} dx + \int_{|u_{n,m}| > 1} |u_{n,m}|(t) dx \leq \int_{\Omega} \Psi(u_{n,m})(t) dx,$$

we conclude that $u_{n,m} \rightarrow 0$ uniformly in t .

Thus $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; L^1(\Omega))$ and passing to the limit in the weak formulation of the approximated problems, one obtains that u is a positive weak solution of problem (40) in $\Omega \times (0, T)$. \square

Next, we see that $\Lambda_{N,s}$ provides a real restriction on λ .

Proposition 5.3. *If $\lambda > \Lambda_{N,s}$, problem (40) has no positive weak supersolution.*

Proof. Consider \tilde{u} as a weak supersolution to the problem

$$(50) \quad \begin{cases} u_t + (-\Delta)^s u - \Lambda_{N,s} \frac{u}{|x|^{2s}} &= (\lambda - \Lambda_{N,s}) \frac{u}{|x|^{2s}} + g \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T]. \end{cases}$$

Since in the left hand side the constant is $\Lambda_{N,s}$, we are in the case $\alpha = 0$, and by Theorem 5.1, necessarily

$$\left((\lambda - \Lambda_{N,s}) \frac{\tilde{u}}{|x|^{2s}} + g \right) |x|^{-\frac{N-2s}{2}} \in L^1(B_r(0) \times (t_1, t_2)),$$

for any $B_r(0) \times (t_1, t_2) \subset \subset \Omega \times (0, T)$ small enough. In particular, this implies

$$(\lambda - \Lambda_{N,s}) \frac{\tilde{u}}{|x|^{2s}} |x|^{-\frac{N-2s}{2}} \in L^1(B_r(0) \times (t_1, t_2)),$$

and hence, applying Corollary 4.12 again,

$$(\lambda - \Lambda_{N,s}) |x|^{-N} \in L^1(B_r(0) \times (t_1, t_2)),$$

what is a contradiction. Therefore, there does not exist a positive supersolution if $\lambda > \Lambda_{N,s}$. \square

Remark 5.4. The previous nonexistence result implies that for $\lambda > \Lambda_{N,s}$ an instantaneous and complete blow up phenomena occurs. The proof is a simple adaptation of Theorem 6.4, where this result is proved for a more involved semilinear problem.

Furthermore, we can state a nonexistence result that shows the optimality of the power $p = 1$ in the singular term $\frac{u^p}{|x|^{2s}}$. The proof for this nonlocal problem closely follows the classical case, due to H. Brezis and X. Cabré (see [14]).

Theorem 5.5. *Let $p > 1$, and let $u \geq 0$ satisfy*

$$u_t + (-\Delta)^s u \geq \frac{u^p}{|x|^{2s}} \text{ in } \Omega \times (0, T),$$

in the weak sense. Then $u \equiv 0$.

Proof. Consider a cylinder $B_\tau(0) \times (t_1, t_2)$. If $u \not\equiv 0$, by the Maximum Principle (Theorem 2.14), we know that there exists $\varepsilon > 0$ so that

$$u \geq \varepsilon \text{ in } B_\tau(0) \times (t_1, t_2).$$

Let define

$$\phi(s) = \begin{cases} \frac{1}{(p-1)\varepsilon^{p-1}} - \frac{1}{(p-1)s^{p-1}} & \text{if } s \geq \varepsilon, \\ \frac{1}{\varepsilon^p}(s - \varepsilon) & \text{if } s < \varepsilon. \end{cases}$$

Notice that $0 \leq \phi < +\infty$ in $[\varepsilon, +\infty)$, $\phi(\varepsilon) = 0$, $\phi'(\varepsilon) = 0$, and ϕ is a \mathcal{C}^1 function satisfying $\phi'(s) = \frac{1}{s^p}$ for $s \geq \varepsilon$. Moreover, since ϕ is concave, it follows that $(-\Delta)^s(\phi(u)) \geq \phi'(u)(-\Delta)^s u$ and thus,

$$(\phi(u))_t + (-\Delta)^s(\phi(u)) \geq \phi'(u)(u_t + (-\Delta)^s u) \geq \frac{1}{|x|^{2s}} \text{ in } B_\tau(0) \times (t_1, t_2)$$

with $u \geq \varepsilon$.

Without loss of generality we can assume that $\tau = 1$. Define $w(x, t) = (t - t_1)\vartheta(x)$ where

$$\vartheta(x) = \begin{cases} \log\left(\frac{1}{|x|}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

then $w(x, t_1) = 0$ in $B_1(0)$ and $w(x, t) = 0$ in $\mathbb{R}^N \setminus B_1(0)$.

We claim that

$$w_t + (-\Delta)^s w \leq \frac{C}{|x|^{2s}} \text{ in } B_1(0) \times (t_1, t_2)$$

where $C \equiv C(t_1, t_2) > 0$. Indeed, we have

$$w_t + (-\Delta)^s w = \vartheta(x) + (t - t_1)(-\Delta)^s \vartheta.$$

It is clear that $\vartheta(x) \leq \frac{C_1}{|x|^{2s}}$ in $B_1(0) \times (t_1, t_2)$, and hence to prove the claim we have to show that

$$(-\Delta)^s \vartheta(x) \leq \frac{C_2}{|x|^{2s}} \text{ for all } x \in B_1(0).$$

In fact,

$$\begin{aligned} (-\Delta)^s \vartheta(x) &= \int_{\mathbb{R}^N} \frac{(\vartheta(x) - \vartheta(y))}{|x - y|^{N+2s}} dy \\ &= \int_{\{|y| < 1\}} \frac{(\vartheta(x) - \vartheta(y))}{|x - y|^{N+2s}} dy + \int_{\{|y| > 1\}} \frac{\vartheta(x)}{|x - y|^{N+2s}} dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

We closely follow the arguments in [26] to estimate the integrals above. By setting $r := |x|$ and $\rho := |y|$, then $x = rx'$, $y = \rho y'$ where $|x'| = |y'| = 1$. Thus,

$$I_1(x) = \int_0^1 \log\left(\frac{\rho}{r}\right) \rho^{N-1} \left(\int_{|y'|=1} \frac{dH^{N-1}(y')}{|rx' - \rho y'|^{N+2s}} \right) d\rho.$$

Calling $\sigma := \frac{\rho}{r}$, then

$$I_1(x) = \frac{D_1(|x|)}{|x|^{2s}},$$

where

$$D_1(r) = \int_0^{\frac{1}{r}} \log(\sigma) \sigma^{N-1} K(\sigma) d\sigma,$$

and

$$K(\sigma) := \int_{|y'|=1} \frac{dH^{N-1}(y')}{|x' - \sigma y'|^{N+2s}} = 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-2}(\eta)}{(1 - 2\sigma \cos(\eta) + \sigma^2)^{\frac{N+2s}{2}}} d\eta.$$

In the same way we have

$$I_2(x) = \frac{D_2(|x|)}{|x|^{2s}} \text{ where } D_2(r) = -\log(r) \int_{\frac{1}{r}}^{+\infty} \sigma^{N-1} K(\sigma) d\sigma.$$

Combining the estimates above, we get

$$(-\Delta)^s \vartheta(x) = \frac{D(|x|)}{|x|^{2s}}$$

with

$$D(r) := D_1(r) + D_2(r) = \int_0^{\frac{1}{r}} \log(\sigma) \sigma^{N-1} K(\sigma) d\sigma - \log(r) \int_{\frac{1}{r}}^{+\infty} \sigma^{N-1} K(\sigma) d\sigma.$$

Notice that $K(\sigma) \leq C|1 - \sigma|^{-1-2s}$ as $\sigma \rightarrow 1$ and $K\left(\frac{1}{\sigma}\right) = \sigma^{N+2s} K(\sigma)$ for all $\sigma > 0$.

If $s \leq \frac{1}{2}$, then using the behavior of K at $+\infty$ we can easily prove that $|D_1(r)| + |D_2(r)| \leq C$ for all $r \leq 1$. To study the general case we need to do some sharp computations. Since

$$D(r) \leq \int_0^{+\infty} \log(\sigma) \sigma^{N-1} K(\sigma) d\sigma = \bar{D},$$

then to finish we have to show that $|\bar{D}| < \infty$. Notice that

$$\bar{D} = \int_0^1 \log(\sigma) \sigma^{N-1} K(\sigma) d\sigma + \int_1^{+\infty} \log(\sigma) \sigma^{N-1} K(\sigma) d\sigma.$$

By putting $\theta := \frac{1}{\sigma}$ in the first integral, and by using the fact that $K\left(\frac{1}{\theta}\right) = \theta^{N+2s} K(\theta)$, it follows that

$$\bar{D} = \int_1^{+\infty} K(\sigma) \log(\sigma) (\sigma^{N-1} - \sigma^{2s-1}) d\sigma.$$

Now, due to the behavior of K near 1 and ∞ we obtain that $0 < \bar{D} < \infty$. Thus

$$(-\Delta)^s \vartheta(x) \leq \frac{\bar{D}}{|x|^{2s}}.$$

Hence

$$w_t + (-\Delta)^s w \leq \frac{C}{|x|^{2s}} \text{ in } B_1(0) \times (t_1, t_2),$$

where $C = C_1 + (t_2 - t_1)\bar{D}$ and the claim follows.

Fixed $\varepsilon > 0$ such that $\varepsilon C < 1$, we obtain that

$$(\phi(u) - \varepsilon w)_t + (-\Delta)^s (\phi(u) - \varepsilon w) \geq 0 \text{ in } B_1(0) \times (t_1, t_2),$$

in the weak sense. Since $(\phi(u) - \varepsilon w)(x, t_1) \geq 0$ in $B_1(0)$ and $(\phi(u) - \varepsilon w)(x, t_1) \geq 0$ in $\mathbb{R}^N \setminus B_1(0) \times (t_1, t_2)$, then the comparison principle implies that $\phi(u) - \varepsilon w \geq 0$ in $B_1(0) \times (t_1, t_2)$. Since w is unbounded, we reach a contradiction with the fact that ϕ is a bounded function, and the proof is finished. \square

Thus, as a straightforward consequence we obtain the following result.

Corollary 5.6. *Let $g \in L^1(\Omega \times (0, T))$, $g \geq 0$, and $p > 1$. Therefore, the problem*

$$\begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u^p}{|x|^{2s}} + g \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \geq 0 \text{ if } x \in \Omega, \end{cases}$$

has no positive weak solution.

6. EXISTENCE AND NONEXISTENCE RESULTS FOR A SEMILINEAR PROBLEM

The goal of this section is to study how the addition of a semilinear term of the form u^p , with $p > 1$, interferes with the solvability of the previous problems. As in the classical heat equation, see [3], the relevant feature is that for every $0 < \lambda < \Lambda_{N,s}$ there exists a threshold for the existence, $p_+(\lambda, s)$, that depends on the spectral parameter. Indeed, we will consider the problem

$$(51) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p + f \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

with $p > 1$ and $0 < \lambda < \Lambda_{N,s}$. By weak or energy solutions of this problem, we mean solutions in the sense of Definition 2.4 and Definition 2.5 by fixing $F = \lambda \frac{u}{|x|^{2s}} + u^p + cf$.

We will prove that there exists such critical exponent $p_+(\lambda, s)$ so that one can prove existence of solution for problem (51) whether $1 < p < p_+(\lambda, s)$, and nonexistence for $p > p_+(\lambda, s)$. Following the same ideas as in [10], one can expect $p_+(\lambda, s)$ to depend on s and λ , and in particular to satisfy

$$p_+(\lambda, s) = 1 + \frac{2s}{\frac{N-2s}{2} - \alpha} = 1 + \frac{2s}{\gamma}.$$

Note that if $\lambda = \Lambda_{N,s}$, namely, $\alpha = 0$, then $p_+(\lambda, s) = 2^* - 1$, and if $\lambda = 0$, i.e., $\alpha = \frac{N-2s}{2}$, then $p_+(\lambda, s) = \infty$.

We will need some auxiliary results that allow us to build a solution whenever we have a supersolution. To prove existence of a weak solution to (51) with L^1 data from a weak supersolution, we will consider the *solution obtained as limit of solutions of approximated problems* (see for instance [20] in the local parabolic operators case).

Lemma 6.1. *If $\bar{u} \in \mathcal{C}([0, T]; L^1(\Omega))$ is a weak positive supersolution to the equation in (51) with $\lambda \leq \Lambda_{N,s}$ and $f \in L^1(\Omega \times (0, T))$, then there exists a positive minimal weak solution to problem (51) obtained as limit of solutions of approximated problems.*

Proof. If \bar{u} is a positive supersolution to (51) with $\lambda \leq \Lambda_{N,s}$, we construct a sequence $\{u_n\}_{n \in \mathbb{N}}$ starting with

$$(52) \quad \begin{cases} u_{0t} + (-\Delta)^s u_0 &= T_1(f) \text{ in } \Omega \times (0, T), \\ u_0(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u_0(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u_0(x, 0) &= T_1(u_0(x)) \text{ if } x \in \Omega. \end{cases}$$

By the Weak Comparison Principle (Lemma 2.9), it follows that $u_0 \leq \bar{u}$ in $\mathbb{R}^N \times (0, T)$. By iteration we define

$$(53) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n &= \lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + u_{n-1}^p + T_n(f) \text{ in } \Omega \times (0, T), \\ u_n(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u_n(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u_n(x, 0) &= T_n(u_0(x)) \text{ if } x \in \Omega. \end{cases}$$

In fact, $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{C}([0, T]; L^1(\Omega)) \cap L^p([0, T]; L^p(\Omega))$ (see [34]). As above it follows that $u_0 \leq \dots \leq u_{n-1} \leq u_n \leq \bar{u}$ in $\mathbb{R}^N \times (0, T)$, so we obtain the pointwise limit $u := \lim u_n$ that verifies $u \leq \bar{u}$ and

$$(54) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p + f \text{ in } \Omega \times (0, T), \\ u(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

in the weak sense. The fact that $u \in \mathcal{C}([0, T]; L^1(\Omega))$ follows as in the proof of Theorem 5.2. \square

Likewise, if the supersolution belongs to the energy space, the solution we find will be also an energy solution.

Lemma 6.2. *If $\bar{u} \in L^2(0, T; H_0^s(\Omega))$ with $\bar{u}_t \in L^2(0, T; H^{-s}(\Omega))$ is a positive finite energy supersolution to (51) with $\lambda \leq \Lambda_{N,s}$ and $f \in L^2(0, T; H^{-s}(\Omega))$, then there exists a positive minimal energy solution to problem (51) obtained as limit of solutions of the approximated problems.*

Proof. Proceeding as in the proof of Lemma 6.1 and using the Comparison Principle for energy solutions (Lemma 2.11), we can build a sequence $\{u_n\}_{n \in \mathbb{N}}$ of energy solutions of the approximated problems (53), so that,

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq \bar{u} \quad \text{in } \mathbb{R}^N \times (0, T).$$

Hence, by the Monotone Convergence Theorem we can define $u := \lim_{n \rightarrow \infty} u_n \leq \bar{u}$. Moreover, applying the energy formulation of u_n ,

$$\begin{aligned}
\|u_n\|_{L^2(0,T;H_0^s(\Omega))}^2 &= \int_0^T \|u_n(\cdot, t)\|_{H_0^s(\Omega)}^2 dt = \int_0^T \int_Q \frac{(u_n(x, t) - u_n(y, t))^2}{|x - y|^{N+2s}} dx dy dt \\
&= \frac{2}{a_{N,s}} \left\{ \int_0^T \int_\Omega \left(\lambda \frac{u_n^2}{|x|^{2s}} + u_n^{p+1} + T_n(f)u_n \right) dx dt - \int_0^T \int_\Omega (u_n)_t u_n dx dt \right\} \\
&= \frac{2}{a_{N,s}} \left\{ \int_0^T \int_\Omega \left(\lambda \frac{u_n^2}{|x|^{2s}} + u_n^{p+1} + T_n(f)u_n \right) dx dt - \frac{1}{2} \int_\Omega u_n(x, T)^2 dx \right. \\
&\quad \left. + \frac{1}{2} \int_\Omega u_n(x, 0)^2 dx \right\} \\
&\leq \frac{2}{a_{N,s}} \left\{ \int_0^T \int_\Omega \left(\lambda \frac{\bar{u}^2}{|x|^{2s}} + \bar{u}^{p+1} + f\bar{u} \right) dx dt + \frac{1}{2} \int_\Omega \bar{u}(x, 0)^2 dx \right\} \\
&\leq C.
\end{aligned}$$

Thus, up to a subsequence, we know that $u_n \rightharpoonup u$ in $L^2(0, T; H_0^s(\Omega))$. Likewise, for every $0 \leq t \leq T$,

$$\begin{aligned}
\|(u_n)_t\|_{H^{-s}(\Omega)} &= \sup_{\|\varphi\|_{H_0^s(\Omega)} \leq 1} \left| \int_\Omega (u_n)_t \varphi dx \right| \\
&\leq \sup_{\|\varphi\|_{H_0^s(\Omega)} \leq 1} \left| \int_Q \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \right| \\
&\quad + \sup_{\|\varphi\|_{H_0^s(\Omega)} \leq 1} \left\{ \int_\Omega u_n^p \varphi dx + \lambda \int_\Omega \frac{u_n \varphi}{|x|^{2s}} dx + \int_\Omega T_n(f) \varphi \right\} \\
&\leq \sup_{\|\varphi\|_{H_0^s(\Omega)} \leq 1} \left\{ \|u_n\|_{H_0^s(\Omega)} \|\varphi\|_{H_0^s(\Omega)} + \int_\Omega \bar{u}^p \varphi dx + \lambda \int_\Omega \frac{\bar{u} \varphi}{|x|^{2s}} dx + \int_\Omega f \varphi dx \right\} \\
&\leq C(\|u_n\|_{H_0^s(\Omega)} + 1 + \|f\|_{L^2(\Omega)}).
\end{aligned}$$

Hence,

$$\int_0^T \|(u_n)_t\|_{H^{-s}(\Omega)}^2 dt \leq C(\|u_n\|_{L^2(0,T;H_0^s(\Omega))}^2 + 1 + \|f\|_{L^2(0,T;L^2(\Omega))}^2) \leq C,$$

and therefore, up to a subsequence, $(u_n)_t \rightharpoonup u_t$ in $L^2(0, T; H^{-s}(\Omega))$, and we can pass to the limit to conclude that u is a finite energy solution to (51). \square

6.1. Nonexistence results for $p > p_+(\lambda, s)$. Instantaneous and complete blow up. Assume first that p is greater than the threshold exponent $p_+(\lambda, s)$. Thus, we can formulate the nonexistence result as follows.

Theorem 6.3. *Let $\lambda \leq \Lambda_{N,s}$. If $p > p_+(\lambda, s)$, then problem (51) has no positive weak supersolution. In the case where $f \equiv 0$, the unique nonnegative supersolution is $u \equiv 0$.*

Proof. Without loss of generality, we can assume $f \in L^\infty(\Omega \times (0, T))$. We argue by contradiction. Assume that \tilde{u} is a positive weak supersolution of (51). Then $\tilde{u}_t + (-\Delta)^s \tilde{u} - \lambda \frac{\tilde{u}}{|x|^2} \geq 0$ in $\Omega \times (0, T)$ in the weak sense.

Since \tilde{u} is also a weak supersolution in any $B_R(0) \times (T_1, T_2) \subset\subset \Omega \times (0, T)$, then by Lemma 6.1, the problem

$$(55) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p + f \text{ in } B_R(0) \times (T_1, T_2), \\ u(x, t) &> 0 \text{ in } B_R(0) \times (T_1, T_2), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus B_R(0)) \times [T_1, T_2], \\ u(x, T_1) &= \tilde{u}(x, T_1) \text{ if } x \in B_R(0), \end{cases}$$

has a minimal solution u obtained by approximation of truncated problems in $B_R(0) \times (T_1, T_2)$. In particular $u = \lim u_n$, with $u_n \in L^\infty(B_R(0) \times (T_1, T_2))$ being the energy solution to (53) in $B_R(0) \times (T_1, T_2)$.

Notice that $u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \geq 0$ in $B_{r_1}(0) \times (T_1, T_2)$ in the weak sense, and therefore, by Corollary 4.12, for any cylinder $B_r(0) \times (t_1, t_2)$, with $0 < r < r_1 < R$, $0 < T_1 < t_1 < t_2 < T_2 \leq T$ there exists a constant $C = C(N, r_1, t_1, t_2)$ such that $u \geq C|x|^{-\frac{N-2s}{2}+\alpha}$ and then for r small enough $u > 1$ in $B_r(0) \times (T_1, T_2)$.

In particular, since $u \in L^1(\Omega \times (0, T))$, then using the fact that $u \geq 1$ in $B_r(0) \times (T_1, T_2)$, we reach that $\log(u) \in L^p(B_r(0) \times (t_1, t_2))$, for all $p \geq 1$. By a suitable scaling, we can assume that the cylinder is $B_r(0) \times (0, \tau)$.

Let $\phi \in C_0^\infty(B_r(0))$, then using $\frac{|\phi|^2}{u_n}$ as a test function in the approximated problems (53) and applying the Picone (Theorem B.4) and Sobolev (Theorem 2.1) inequalities,

$$\begin{aligned} \int_0^\tau \int_{B_r(0)} u_n^{p-1} \phi^2 dx dt &\leq \int_0^\tau \int_{B_r(0)} \frac{|\phi|^2}{u_n} u_{nt} dx dt + \int_0^\tau \int_{B_r(0)} (-\Delta)^s u_n \frac{|\phi|^2}{u_n} dx dt \\ &\leq \int_{B_r(0)} |\log u_n(x, \tau)| \phi^2 dx + C'(N, s, \tau) \|\phi\|_{H_0^s(\Omega)}^2. \end{aligned}$$

Therefore, passing to the limit as n tends to infinity, and considering that $u \geq C|x|^{-\frac{N-2s}{2}+\alpha}$ in $B_r(0) \times (0, \tau)$, we obtain

$$\begin{aligned} \int_{B_r(0)} |\log u(x, \tau)| \phi^2 dx + C'(N, s, \tau) \|\phi\|_{H_0^s(\Omega)}^2 &\geq \int_0^\tau \int_{B_r(0)} u^{p-1} \phi^2 dx dt \\ &\geq C \int_0^\tau \int_{B_r(0)} \frac{\phi^2}{|x|^{(p-1)(\frac{N-2s}{2}-\alpha)}} dx dt. \end{aligned}$$

Using Hölder and Sobolev inequalities, it follows that

$$\begin{aligned} \int_{B_r(0)} |\log(u(x, \tau))| |\phi|^2 dx &\leq \left(\int_{B_r(0)} |\phi|^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{B_r(0)} |\log u(x, \tau)|^{\frac{N}{2s}} dx \right)^{\frac{2s}{N}} \\ &\leq \left(\int_{B_r(0)} |\log u(x, \tau)|^{\frac{N}{2s}} dx \right)^{\frac{2s}{N}} S \|\phi\|_{H_0^s(\Omega)}^2, \end{aligned}$$

where S is the optimal constant in the Sobolev embedding. Thus we have

$$\left[C'(N, s, \tau) + \left(\int_{B_r(0)} |\log u(x, \tau)|^{\frac{N}{2s}} dx \right)^{\frac{2s}{N}} S \right] \|\phi\|_{H_0^s(\Omega)}^2 \geq C \int_{B_r(0)} \frac{\phi^2}{|x|^{(p-1)(\frac{N-2s}{2}-\alpha)}} dx.$$

Since $p > p_+(\lambda, s)$ then, for a cylinder small enough, $(p-1) \left(\frac{N-2s}{2} - \alpha \right) > 2s$ and we obtain a contradiction with the Hardy inequality. \square

The previous nonexistence result is very strong in the sense that a complete and instantaneous blow up phenomenon occurs. That is, if u_n is the solution to the approximated problems (53), then $u_n(x, t) \rightarrow \infty$ as $n \rightarrow \infty$, where (x, t) is an arbitrary point in $\Omega \times (0, T)$.

Theorem 6.4. *Let u_n be a solution to the problem (53) with $p > p_+(\lambda, s)$. Then $u_n(x_0, t_0) \rightarrow \infty$, for all $(x_0, t_0) \in \Omega \times (0, T)$.*

Proof. Without loss of generality, we can assume that $\lambda \leq \Lambda_N$. The existence of a positive solution to problem (53) is clear and, due to the Comparison Principle, we know that $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$.

Suppose by contradiction that there exists $(x_0, t_0) \in \Omega \times (0, T)$ such that

$$u_n(x_0, t_0) \rightarrow C_0 < \infty \text{ as } n \rightarrow \infty.$$

By using the Harnack inequality (see Lemma 2.13), there exists $s_0 > 0$ and a positive constant $C = C(N, s_0, t_0, \beta)$ such that

$$\iint_{R_0^-} u_n(x, t) dx dt \leq C \operatorname{ess\,inf}_{R_0^+} u_n \leq C,$$

where $R_0^- = B_{s_0}(x_0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R_0^+ = B_{s_0}(x_0) \times (t_0 + \frac{1}{4}\beta, t_0 + \frac{3}{4}\beta)$.

Without loss of generality, we can suppose $x_0 = 0$. Otherwise, we can find a finite sequence of points $\{x_i\}_{i=0}^M$, ending with $x_M = 0$, and of radius $\{s_i\}_{i=0}^M$ such that $B_{s_i}(x_i) \subset \Omega$, $B_{s_i}(x_i) \cap B_{s_{i+1}}(x_{i+1}) \neq \emptyset$, for all $i = 0, \dots, M$ and, by the Harnack inequality,

$$\iint_{R_i^-} u_n(x, t) dx dt \leq C \operatorname{ess\,inf}_{R_i^+} u_n,$$

where $R_i^- = B_{s_i}(x_i) \times (t_i - \frac{3}{4}\beta, t_i - \frac{1}{4}\beta)$ and $R_i^+ = B_{s_i}(x_i) \times (t_i + \frac{1}{4}\beta, t_i + \frac{3}{4}\beta)$, $t_i \in (0, T)$ and β is small enough so that $t_i - \frac{3}{4}\beta > 0$ and $t_i + \frac{3}{4}\beta < T$ for all $i = 0, \dots, M$. Let us choose now $t_i = t_{i-1} - \beta$ for $i = 1, \dots, M$. Note that in this case

$$(t_i + \frac{1}{4}\beta, t_i + \frac{3}{4}\beta) = (t_{i-1} - \frac{3}{4}\beta, t_{i-1} - \frac{1}{4}\beta),$$

and in particular, $R_i^+ \cap R_{i-1}^- \neq \emptyset$. Thus,

$$\begin{aligned} \iint_{R_M^-} u_n(x, t) dx dt &\leq \operatorname{ess\,inf}_{R_M^+} u_n(x, t) \leq \operatorname{ess\,inf}_{R_M^+ \cap R_{M-1}^-} u_n(x, t) \\ &\leq \frac{1}{|R_M^+ \cap R_{M-1}^-|} \iint_{R_M^+ \cap R_{M-1}^-} u_n(x, t) dx dt \\ &\leq \frac{1}{|R_M^+ \cap R_{M-1}^-|} \iint_{R_{M-1}^-} u_n(x, t) dx dt \\ &\leq \dots \leq C \iint_{R_0^-} u_n(x, t) dx dt \leq \tilde{C}. \end{aligned}$$

Therefore, supposing $x_0 = 0$, by the Monotone Convergence Theorem there exists $u \geq 0$ such that $u_n \uparrow u$ strongly in $L^1(B_r(0) \times (t_1, t_2))$. Let now φ be the solution to the problem

$$(56) \quad \begin{cases} -\varphi_t + (-\Delta)^s \varphi &= \chi_{B_r(0) \times [t_1, t_2]} \text{ in } \Omega \times (0, T), \\ \varphi(x, t) &> 0 \text{ in } \Omega \times (0, T), \\ \varphi(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T], \\ \varphi(x, T) &= 0 \text{ in } \Omega. \end{cases}$$

Note that, due to the regularity of the right hand sides of problems (53) and (56), both u_n and φ are in the energy space, and thus both can be used as test functions in the energy formulation of

the problems. Indeed, considering first u_n as test function in (56) and then, after integrating by parts, φ in (53), and defining $\eta := \inf_{B_r(0) \times (t_1, t_2)} \varphi(x, t)$, we have

$$\begin{aligned} C &\geq \int_{t_1}^{t_2} \int_{B_r(0)} u_n(x, t) dx dt \\ &\geq \lambda \int_0^T \int_{\Omega} \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} \varphi dx dt + \int_0^T \int_{\Omega} u_{n-1}^p \varphi dx dt + \int_0^T \int_{\Omega} T_n(f) \varphi dx dt \\ &\geq \lambda \eta \int_{t_1}^{t_2} \int_{B_r(0)} \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx dt + \eta \int_{t_1}^{t_2} \int_{B_r(0)} u_{n-1}^p dx dt + \eta \int_{t_1}^{t_2} \int_{B_r(0)} T_n(f) dx dt. \end{aligned}$$

By the Monotone Convergence Theorem,

$$\begin{aligned} \frac{u_{n-1}^p}{|x|^{2s} + \frac{1}{n}} &\rightarrow u^p \text{ in } L^1(B_r(0) \times (t_1, t_2)), \\ \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} &\nearrow \frac{u}{|x|^{2s}} \text{ in } L^1(B_r(0) \times (t_1, t_2)), \\ T_n(f) &\rightarrow f \text{ in } L^1(B_r(0) \times (t_1, t_2)). \end{aligned}$$

Thus it follows that u is a weak supersolution to (1) in $B_r(0) \times (t_1, t_2)$, a contradiction with Theorem 6.3. \square

6.2. Existence results for $1 < p < p_+(\lambda, s)$. The goal now is to consider the complementary interval of powers, $1 < p < p_+(\lambda, s)$, and to prove that under some suitable hypotheses on f and u_0 , problem (51) has a positive solution. We will consider here the case $f \equiv 0$. For the case $f \not\equiv 0$, see Remark 6.6.

First of all, notice that if $0 < \lambda \leq \Lambda_{N,s}$ and $1 < p < p_+(\lambda, s)$, the stationary problem

$$(57) \quad (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

has a positive supersolution w , depending on the following cases:

(A) $0 < \lambda < \Lambda_{N,s}$: In Proposition 2.3 of [10], the authors find a positive solution to the problem

$$(58) \quad (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p + \mu u^q \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

for μ small enough, $0 < q < 1$ and $1 < p < p_+(\lambda, s)$. In particular, for every $\mu \geq 0$ this solution is a supersolution of (57). Note that for $1 < p \leq 2_s^* - 1$ this supersolution is in the energy space, and for $2_s^* - 1 < p < p_+(\lambda, s)$, it is a weak positive supersolution.

(B) If $\lambda = \Lambda_{N,s}$, then $p_+(\lambda, s) = 2_s^* - 1$. Thus, instead of $H_0^s(\Omega)$, we consider the Hilbert space $H(\Omega)$ defined in (10). Since $H(\Omega)$ is compactly embedded in $L^p(\Omega)$ for all $1 \leq p < 2_s^*$, classical variational methods in the space $H(\Omega)$ allow us to prove the existence of a positive solution w to the stationary problem (57).

Theorem 6.5. *Assume that $0 < \lambda \leq \Lambda_{N,s}$ and $1 < p < p_+(\lambda, s)$. Suppose that $u_0(x) \leq \bar{w}$, where \bar{w} is a supersolution to the stationary problem*

$$(-\Delta)^s w = \lambda \frac{w}{|x|^{2s}} + w^p \text{ in } \Omega, \quad w(x) > 0 \text{ in } \Omega, \quad w(x) = 0 \text{ on } \mathbb{R}^N \setminus \Omega.$$

Then for all $T > 0$, the problem

$$(59) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p \text{ in } \Omega \times (0, T), \\ u(x, t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ u(x, 0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

has a global positive solution. If \bar{w} is a weak supersolution, the solution will be also weak, and likewise, if \bar{w} is an energy supersolution, problem (59) will have an energy solution.

Proof. Since $\bar{w}(x) \geq u_0(x)$ for all $x \in \Omega$, then \bar{w} is a positive supersolution to problem (59). Hence, we conclude just by applying Lemma 6.1, whether \bar{w} is a weak supersolution, or Lemma 6.2, if \bar{w} is an energy supersolution. \square

Remark 6.6.

(I) With the results above we find the optimality of the power $p_+(\lambda, s)$, what was our main aim. Nevertheless, it could be interesting to know the optimal class of data for which there exists a solution and the regularity of such solutions according to the regularity of the data. In this direction, considering $g = u^p + cf$ in problem (40), Theorem 5.1 establishes that, necessarily,

$$\int_{B_r(0)} |x|^{-\frac{N-2s}{2}+\alpha} u_0 dx < +\infty,$$

if we expect to find a solution of problem (51).

(II) In the presence of a source term $f \geq 0$, if $f(x, t) \leq \frac{c_0(t)}{|x|^{2s}}$ with $c_0(t)$ bounded and *sufficiently small*, then the computation above allows us to prove the existence of a supersolution. Then the existence of a minimal solution to problem (1) follows for all $p < p_+(\lambda, s)$.

APPENDIX A. REGULARITY IN BOUNDED DOMAINS: RELATION BETWEEN WEAK SOLUTIONS AND VISCOSITY SOLUTIONS

The regularity of the solutions to the problem

$$(60) \quad \begin{cases} u_t + (-\Delta)^s u &= f(x, t) \text{ in } \mathbb{R}^N \times (0, T), \\ u(x, 0) &= u_0(x) \text{ if } x \in \mathbb{R}^N, \end{cases}$$

according to the regularity of the data can be seen in [15]. In this situation the properties of the fundamental solution play a crucial role.

We will try to prove, in particular, that the class of test functions,

$$\mathcal{T} := \left\{ \phi : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}, \text{ s.t. } \begin{cases} \phi_t + (-\Delta)^s \phi = \varphi, \varphi \in L^\infty(\Omega \times (0, T)), \\ \phi = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0, T], \quad \phi(x, 0) = 0 \text{ in } \Omega \end{cases} \right\},$$

is a class of *regular functions* for which the equation is also verified in an almost everywhere way. In this sense we will say that the functions in \mathcal{T} are classical or strong solutions to the fractional heat equation.

To start we will consider such a function ϕ as an energy solution to the fractional heat equation. According to the results in [34] we find that every $\phi \in \mathcal{T}$ belongs in particular to $L^\infty(\Omega \times (0, T))$ and by using the results in [24] we find that ϕ is Hölder continuous in $\Omega \times (0, T)$.

Moreover in [25] the authors prove the following sharper result.

Theorem A.1. *Assume that Ω is a $C^{1,1}$ bounded domain in \mathbb{R}^N and let $v_0 \in L^2(\Omega)$. Consider v the unique weak solution to problem*

$$(61) \quad \begin{cases} v_t + (-\Delta)^s v &= 0 & \text{in } \Omega \times (0, \infty) \\ v &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \times (0, \infty) \\ v(x, 0) &= v_0(x) & \text{in } \Omega. \end{cases}$$

Then, for each $\epsilon > 0$,

i)

$$(62) \quad \sup_{t > \epsilon} \|u(\cdot, t)\|_{C^s} \leq C_1(\epsilon) \|v_0\|_{L^2(\Omega)}.$$

ii)

$$(63) \quad \sup_{t > \epsilon} \left\| \frac{u(\cdot, t)}{\delta^s} \right\|_{C^{s-\eta}} \leq C_2(\epsilon) \|v_0\|_{L^2(\Omega)} \quad \text{for any } \eta > 0.$$

Both constants C_1 and C_2 blow up when $\epsilon \rightarrow 0$.

In addition, for every $j \in \mathbb{N}$,

$$(64) \quad \sup_{t > \epsilon} \left\| \frac{\partial^j u(\cdot, t)}{\partial t^j} \right\|_{C^s} \leq C_j(\epsilon) \|v_0\|_{L^2(\Omega)}.$$

The proof in [25] is a consequence of the regularity up to the boundary of the normalized eigenfunction of the elliptic Dirichlet problem and the use of separation of variables.

As a consequence of the previous result we have the following extension property.

Corollary A.2. *Let v be as in Theorem A.1, then v can be extended as a continuous function to $\mathbb{R}^N \times (0, \infty)$.*

We refer to [17] (Definition 2.8) for the concept of viscosity solution (and the definition of the class S of test functions), and to Theorem 5.3 of the same article to prove the stability of viscosity solutions by uniform limits. Therefore, since we are working in the linear setting, via a convenient mollification we can assume that v is uniform limit on compact sets of viscosity solutions for approximated problems and thus it is a viscosity solution.

Hence, for a fixed $(x_0, t_0) \in \Omega \times (0, T)$, we can assume that v is regular in a neighborhood of (x_0, t_0) .

We prove that v is a viscosity subsolution (and in a similar way we prove that it is a supersolution). Consider a test function $\phi \in S$ (see [17] to see the definition of S) such that

- i) $v(x_0, t_0) = \phi(x_0, t_0)$,
- ii) $v(y, s) < \phi(y, s)$.

Then

$$\begin{aligned} \phi_{t-}(x_0, t_0) + (-\Delta)^s \phi(x_0, t_0) &= \lim_{h \rightarrow 0^+} \frac{\phi(x_0, t_0) - \phi(x_0, t_0 - h)}{h} + \int_{\mathbb{R}^N} \frac{\phi(x_0, t_0) - \phi(y, t_0)}{|x - y|^{N+2s}} dy \\ &\leq \lim_{h \rightarrow 0^+} \frac{v(x_0, t_0) - v(x_0, t_0 - h)}{h} + \int_{\mathbb{R}^N} \frac{v(x_0, t_0) - v(y, t_0)}{|x - y|^{N+2s}} dy = 0 \end{aligned}$$

Hence, summarizing, we obtain that v is a viscosity subsolution and in a similar way supersolution.

As an application of the previous result for $f \in L^\infty(\Omega \times (0, T)) \cap C^{\alpha, \beta}(\Omega \times (0, T))$, we get a similar result for the problem

$$(65) \quad \begin{cases} u_t + (-\Delta)^s u &= f(x, t) & \text{in } \Omega \times (0, \infty) \\ u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, \infty) \\ u(x, 0) &= 0 & \text{in } \Omega. \end{cases}$$

Indeed, for fixed $\tau > 0$ we solve

$$(66) \quad \begin{cases} v_t(x, t, \tau) + (-\Delta)^s v(x, t, \tau) &= 0 & \text{in } \Omega \times (\tau, \infty) \\ v(x, t, \tau) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (\tau, \infty) \\ v(x, \tau, \tau) &= f(x, \tau) & \text{in } \Omega. \end{cases}$$

By separation of variables as in [25], we obtain

$$v(x, t, \tau) = \sum_0^\infty c_k(\tau) e_k(x) e^{-\lambda_k(t-\tau)}$$

where (λ_k, e_k) are the eigenvalues and the normalized eigenfunction respectively of the *fractional laplacian* and the Fourier coefficients are defined by

$$c_k(\tau) = \int_\Omega f(y, \tau) e_k(y) dy.$$

Then as above, $v(x, t, \tau)$ can be extended by continuity to the whole space $\mathbb{R}^N \times (\tau, \infty)$.

It is easy to check that

$$u(x, t) := \int_0^t v(x, t, \tau) d\tau,$$

is the extension to $\mathbb{R}^N \times (0, \infty)$ of the unique solution to (65). By mollification and by the stability of viscosity solutions by uniform limits in compact sets, we can assume u regular in the point where we want to test. As before, assume $\phi \in S$ verifying

- i) $u(x_0, t_0) = \phi(x_0, t_0)$,
- ii) $u(y, s) < \phi(y, s)$.

Then

$$\begin{aligned} \phi_{t-}(x_0, t_0) + (-\Delta)^s \phi(x_0, t_0) &= \lim_{h \rightarrow 0^+} \frac{\phi(x_0, t_0) - \phi(x_0, t_0 - h)}{h} + \int_{\mathbb{R}^N} \frac{\phi(x_0, t_0) - \phi(y, t_0)}{|x - y|^{N+2s}} dy \\ &\leq \lim_{h \rightarrow 0^+} \frac{u(x_0, t_0) - u(x_0, t_0 - h)}{h} + \int_{\mathbb{R}^N} \frac{u(x_0, t_0) - u(y, t_0)}{|x - y|^{N+2s}} dy = f(x, t) \end{aligned}$$

Therefore, u is a viscosity subsolution. Analogously, we prove that u is a viscosity supersolution.

Then we can use the regularity results in [16, 17, 32]. In particular, by using Corollaries 2.6. 2.7 in [32], for a second member Hölder continuous in space-time, we find that the equation is verified in the pointwise classical meaning, i.e., is a strong solution in the sense of Definition 1.3 in [11].

APPENDIX B. FUNDAMENTAL INEQUALITIES

We explain in this Appendix the functional results used in the previous sections. Recall first that we defined in (23) the measures μ, ν as

$$d\mu := \frac{dx}{|x|^{2\gamma}}, \quad \text{and} \quad d\nu := \frac{dx dy}{|x|^\gamma |y|^\gamma |x - y|^{N+2s}}.$$

Let begin by the next extension lemma whose proof follows using the same arguments of [6] (see also [21]).

Lemma B.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. Then for all $w \in Y^{s,\gamma}(\Omega)$, there exists $\tilde{w} \in Y^{s,\gamma}(\mathbb{R}^N)$ such that $\tilde{w}|_\Omega = w$ and*

$$\|\tilde{w}\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leq C \|w\|_{Y^{s,\gamma}(\Omega)},$$

where $C := C(N, s, \Omega, \gamma) > 0$.

Recall that $Y_0^{s,\gamma}(\Omega)$ was defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm of $Y^{s,\gamma}(\Omega)$. It is clear that if $\phi \equiv C \in Y_0^{s,\gamma}(\Omega)$, then $C \equiv 0$.

If Ω is a bounded regular domain, we can prove the next Poincaré inequality.

Theorem B.2. *There exists a positive constant $C := C(\Omega, N, s, \gamma)$ such that for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$C \int_\Omega \phi^2(x) d\mu \leq \int_\Omega \int_\Omega (\phi(x) - \phi(y))^2 d\nu.$$

Proof. If $\phi \equiv 0$, the inequality follows trivially. Thus, let us define

$$\lambda_1(\Omega) := \inf_{\{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \neq 0\}} \frac{\int_\Omega \int_\Omega (\phi(x) - \phi(y))^2 d\nu}{\int_\Omega \phi^2(x) d\mu}.$$

Hence, to prove the lemma we need to check that $\lambda_1(\Omega) > 0$. We argue by contradiction, that is, let us suppose $\lambda_1(\Omega) = 0$. Then we get the existence of $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$\int_\Omega \phi_n^2(x) d\mu = 1 \text{ and } \int_\Omega \int_\Omega (\phi_n(x) - \phi_n(y))^2 d\nu \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It is clear that $\|\phi_n\|_{Y^{s,\gamma}(\Omega)} \leq C$, and hence we reach the existence of $\bar{\phi} \in Y^{s,\gamma}(\Omega)$ such that $\phi_n \rightharpoonup \bar{\phi}$ weakly in $Y^{s,\gamma}(\Omega)$.

From the Sobolev inequality in [1] it follows

$$\int_{\Omega} \frac{|\phi_n|^{2_s^*}}{|x|^{2_s^*\gamma}} dx \leq C(N, s, \Omega, \gamma) \|\phi_n\|_{Y^{s,\gamma}(\Omega)} \leq \bar{C},$$

with \bar{C} independent of n .

Using the fact that $Y^{s,\gamma}(\Omega) \subset Y^{s,0}(\Omega)$, it follows from [6] (see also [21]) that $\phi_n \rightarrow \bar{\phi}$ strongly in $L^2(\Omega)$. Hence, combining the estimates above and using Vitali's Lemma we obtain that, up to a subsequence,

$$\phi_n \rightarrow \bar{\phi} \text{ strongly in } L^2(\Omega, d\mu),$$

and thus,

$$(67) \quad \int_{\Omega} \bar{\phi}^2(x) d\mu = 1.$$

Since $\|\bar{\phi}\|_{Y^{s,\gamma}(\Omega)} \leq \|\phi_n\|_{Y^{s,\gamma}(\Omega)}$, taking into consideration that $\int_{\Omega} \int_{\Omega} (\phi_n(x) - \phi_n(y))^2 d\nu \rightarrow 0$ as $n \rightarrow \infty$, we get

$$\phi_n \rightarrow \bar{\phi} \text{ strongly in } Y^{s,\gamma}(\Omega), \text{ thus } \int_{\Omega} \int_{\Omega} (\bar{\phi}(x) - \bar{\phi}(y))^2 d\nu = 0.$$

Hence $\bar{\phi} \equiv 0$. Now, since $\bar{\phi} \in Y_0^{s,\gamma}(\Omega)$, necessarily $\bar{\phi} \equiv 0$, a contradiction with (67). \square

As a direct application of Theorem B.2 we obtain that if Ω is a bounded regular domain, then every $w \in Y_0^{s,\gamma}(\Omega)$ satisfies

$$\|\tilde{w}\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leq C \left(\int_{\Omega} \int_{\Omega} (w(x) - w(y))^2 d\nu \right)^{\frac{1}{2}},$$

where $C := C(N, s, \Omega, \gamma) > 0$ and \tilde{w} is the extension of w given in Lemma B.1.

Define now the operator

$$L_{\gamma,\Omega}(w)(x) := a_{N,s} P.V. \int_{\Omega} (w(x) - w(y)) K(x, y) dy, \text{ where } K(x, y) := \frac{1}{|x|^{\gamma} |y|^{\gamma} |x - y|^{N+2s}}.$$

In the case $\Omega = \mathbb{R}^N$, we have the next result.

Lemma B.3. *If $w(x) := |x|^{-\theta}$, with $0 < \theta < (N - 2s - 2\gamma)$, then there exists a positive constant $C := C(N, s, \gamma, \theta)$ such that*

$$L_{\gamma,\mathbb{R}^N}(w)(x) = C \frac{w(x)}{|x|^{2s+2\gamma}} \text{ a.e. in } \mathbb{R}^N.$$

Proof. In \mathbb{R}^N , the operator has the form

$$L_{\gamma,\mathbb{R}^N}(w)(x) := a_{N,s} P.V. \int_{\mathbb{R}^N} \frac{(w(x) - w(y))}{|x|^{\gamma} |y|^{\gamma} |x - y|^{N+2s}} dy.$$

As in the proof of Theorem 5.5, we closely follow the arguments used in [26].

By setting $r := |x|$ and $\rho := |y|$, then $x = rx'$, and $y = \rho y'$ where $|x'| = |y'| = 1$. Thus,

$$L_{\gamma,\mathbb{R}^N}(w)(x) = \frac{a_{N,s}}{|x|^{\gamma}} \int_0^{+\infty} \frac{(r^{-\theta} - \rho^{-\theta}) \rho^{N-1}}{\rho^{\gamma} r^{N+2s}} \left(\int_{|y'|=1} \frac{dH^{N-1}(y')}{|x' - \frac{\rho}{r} y'|^{N+2s}} \right) d\rho.$$

Set now $\sigma := \frac{\rho}{r}$. Then

$$L_{\gamma, \mathbb{R}^N}(w)(x) = \frac{a_{N,s} w(x)}{|x|^{2s+2\gamma}} \int_0^{+\infty} (1 - \sigma^{-\theta}) \sigma^{N-\gamma-1} \left(\int_{|y'|=1} \frac{dH^{N-1}(y')}{|x' - \sigma y'|^{N+2s}} \right) d\sigma.$$

Define

$$K(\sigma) := \int_{|y'|=1} \frac{dH^{N-1}(y')}{|x' - \sigma y'|^{N+2s}},$$

then

$$(68) \quad K(\sigma) = 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-2}(\eta)}{(1 - 2\sigma \cos(\eta) + \sigma^2)^{\frac{N+2s}{2}}} d\eta.$$

Thus

$$L_{\gamma, \mathbb{R}^N}(w) = \Lambda_{N,s,\gamma} \frac{w(x)}{|x|^{2s+2\gamma}},$$

where

$$\Lambda_{N,s,\gamma} = a_{N,s} \int_0^{+\infty} (\sigma^\theta - 1) \sigma^{N-\gamma-\theta-1} K(\sigma) d\sigma.$$

As in [26], taking into consideration the behavior of K near $\sigma = 1$ and at $+\infty$, we can prove that $|\Lambda_{N,s,\gamma}| < \infty$. To conclude we just have to show that $\Lambda_{N,s,\gamma} > 0$.

Since $K(\frac{1}{s}) = s^{N+2s} K(s)$ for all $s > 0$, we get

$$\begin{aligned} \Lambda_{N,s,\gamma} &= \int_0^1 (\sigma^\theta - 1) \sigma^{N-\gamma-\theta-1} K(\sigma) d\sigma + \int_1^\infty (\sigma^\theta - 1) \sigma^{N-\gamma-\theta-1} K(\sigma) d\sigma \\ &= - \int_1^\infty (\xi^\theta - 1) \xi^{2s+\gamma-1} K(\xi) d\xi + \int_1^\infty (\sigma^\theta - 1) \sigma^{N-\gamma-\theta-1} K(\sigma) d\sigma \\ &= \int_1^\infty K(\sigma) (\sigma^\theta - 1) (\sigma^{N-\gamma-\theta-1} - \sigma^{2s+\gamma-1}) d\sigma. \end{aligned}$$

Since $0 < \theta < N - 2s - 2\gamma$, then the results follows. \square

Next we formulate an extension of a well-known Picone identity, that in the case of regular functions and the Laplacian operator was obtained by Picone in [38] (see [2] for an integral extension related to positive Radon measures).

Theorem B.4. (*Picone's Type Inequality*). Consider $u, v \in H_0^s(\Omega)$, where $(-\Delta)^s u = \tilde{\nu}$ is a bounded Radon measure in Ω , and $u \geq 0$. Then,

$$(69) \quad \int_\Omega \frac{(-\Delta)^s u}{u} v^2 dx \leq \frac{a_{N,s}}{2} \|v\|_{H_0^s(\Omega)}^2.$$

See [34] for a proof. It is worthy to point out that the proof relies in a pointwise inequality. Therefore, we can reformulate the Picone inequality as follows.

Corollary B.5. Let $w \in Y^{s,\gamma}(\Omega)$ be such that $w > 0$ in Ω . Assume that $L_{\gamma,\Omega}(w) = \tilde{\nu}$ with $\tilde{\nu} \in L_{loc}^1(\mathbb{R}^N)$ and $\tilde{\nu} \geq 0$, then for all $u \in C_0^\infty(\Omega)$, we have

$$(70) \quad \frac{a_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^\gamma |y|^\gamma} \geq \langle L_{\gamma,\Omega}(w), \frac{u^2}{w} \rangle_{Y_0^{s,\gamma}(\Omega)}.$$

As a consequence we get the next Hardy type inequality.

Theorem B.6. *There exists a positive constant $C(N, s, \gamma)$ such that for all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have*

$$C \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\phi(x) - \phi(y))^2 d\nu.$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ and define $w(x) := |x|^{-\theta}$, with $0 < \theta < \frac{N-2s-2\gamma}{2}$. Then, by Lemma B.3,

$$L_{\gamma, \mathbb{R}^N}(w)(x) = C \frac{w(x)}{|x|^{2s+2\gamma}} \quad \text{a.e. in } \mathbb{R}^N.$$

Using (70) it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^\gamma |y|^\gamma} \geq \langle L_{\gamma, \mathbb{R}^N}(w), \frac{\phi^2}{w} \rangle_{Y_0^{s, \gamma}(\Omega)} = C(N, s, \gamma) \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx.$$

Hence we conclude. \square

In the case where Ω is a bounded domain, we have:

Theorem B.7. *There exists a positive constant $C(\Omega, N, s, \gamma)$ such that for all $\phi \in C_0^\infty(\Omega)$, we have*

$$C \int_{\Omega} \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx \leq \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu.$$

Proof. Let $\phi \in C_0^\infty(\Omega)$ and define $\tilde{\phi}$ to be the extension of ϕ to \mathbb{R}^N given in Lemma B.1. Then from Theorem B.6, we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^\gamma |y|^\gamma} \geq C(N, s, \gamma) \int_{\mathbb{R}^N} \frac{\tilde{\phi}^2(x)}{|x|^{2s+2\gamma}} dx.$$

Now, using the fact that $\tilde{\phi}|_{\Omega} = \phi$ and combining the results of Lemma B.1 and Theorem B.2, we reach the desired result. \square

In the case of nonzero boundary conditions, we obtain the following version of the Hardy inequality.

Theorem B.8. *There exists a positive constant $C(\Omega, N, s, \gamma)$ such that for all $\phi \in Y^{s, \gamma}(\Omega)$, we have*

$$C \int_{\Omega} \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx \leq \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu + \int_{\Omega} \phi^2(x) d\mu.$$

Proof. Fix $\phi \in Y^{s, \gamma}(\Omega)$ and define $\tilde{\phi}$ as the extension of ϕ to \mathbb{R}^N given in Lemma B.1. Then from Theorem B.6, we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^\gamma |y|^\gamma} \geq C(N, s, \gamma) \int_{\mathbb{R}^N} \frac{\tilde{\phi}^2(x)}{|x|^{2s+2\gamma}} dx \geq C(N, s, \gamma) \int_{\Omega} \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx.$$

Since $\|\tilde{\phi}\|_{Y^{s, \gamma}(\mathbb{R}^N)} \leq C(\Omega) \|\phi\|_{Y^{s, \gamma}(\Omega)}$, then the result follows. \square

Notice now that when $\phi \in C_0^\infty(\Omega)$, the following Sobolev inequality holds (see [1]),

$$(71) \quad \left(\int_{\Omega} \frac{|\phi(x)|^{2_s^*}}{|x|^{\gamma 2_s^*}} dx \right)^{\frac{2}{2_s^*}} \leq C(\Omega, N, s, \gamma) \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu.$$

Moreover, in the particular case $\phi \in Y^{s, \gamma}(B_R)$, as an application of Theorem B.8 we can prove the following improved inequality.

Theorem B.9. *Let $R > 0$ and $\phi \in Y^{s, \gamma}(B_R)$. Then, there exists $C := C(N, s, R, \gamma) > 0$ such that*

$$(72) \quad C \left(\int_{B_R} \frac{|\phi|^{2_s^*}}{|x|^{\gamma 2_s^*}} dx \right)^{\frac{2}{2_s^*}} \leq \int_{B_R} \int_{B_R} (\phi(x) - \phi(y))^2 d\nu + R^{-2s} \int_{B_R} \phi^2 d\mu.$$

PROOF. We prove the result for $R = 1$, and then (72) follows by a scaling argument. We set $\phi_1(x) := \frac{\phi(x)}{|x|^\gamma}$. Then from [6] we know that

$$(73) \quad C \left(\int_{B_1} |\phi_1|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq \int_{B_1} \int_{B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x - y|^{N+2s}} dx dy + \int_{B_1} \phi_1^2 dx.$$

To get the desired result we just have to estimate the term

$$\int_{B_1} \int_{B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x - y|^{N+2s}} dx dy.$$

Since

$$(\phi_1(x) - \phi_1(y))^2 = \frac{(\phi(x) - \phi(y))^2}{|x|^\gamma |y|^\gamma} + \left(\frac{\phi^2(x)}{|x|^\gamma} - \frac{\phi^2(y)}{|y|^\gamma} \right) \left(\frac{1}{|x|^\gamma} - \frac{1}{|y|^\gamma} \right),$$

it follows that

$$\int_{B_1} \int_{B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x - y|^{N+2s}} dx dy \leq \int_{B_1} \int_{B_1} \frac{(\phi(x) - \phi(y))^2}{|x|^\gamma |y|^\gamma |x - y|^{N+2s}} dx dy + \int_{B_1} L_{0,B_1}(|x|^{-\gamma}) \frac{\phi^2(x)}{|x|^\gamma} dx.$$

Proceeding as in the proof of Lemma B.3, since $0 < \gamma < \frac{N-2s}{2}$, we can prove that $L_{0,B_1}(|x|^{-\gamma}) \leq \frac{C}{|x|^{\gamma+2s}}$, and hence

$$\int_{B_1} \int_{B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x - y|^{N+2s}} dx dy \leq \int_{B_1} \int_{B_1} \frac{(\phi(x) - \phi(y))^2}{|x|^\gamma |y|^\gamma |x - y|^{N+2s}} dx dy + C \int_{B_1} \frac{\phi^2}{|x|^{2s+2\gamma}} dx.$$

Finally, using Theorem B.8 and substituting $\phi(x) = |x|^\gamma \phi_1(x)$, we reach (72). \square

We state now a weighted version of the Poincaré-Wirtinger inequality used in the proof of Lemma 4.11.

Theorem B.10. *Let $w \in Y^{s,\gamma}(B_1)$ and assume that ψ is a radial decreasing function such that $\text{supp } \psi \subset B_1$ and $0 \leq \psi \leq 1$. Define*

$$W_\psi := \frac{\int_{B_1} w(x) \psi(x) d\mu}{\int_{B_1} \psi(x) d\mu}.$$

Then, there exists $C := C(N, s, \psi) > 0$ such that

$$\int_{B_1} (w(x) - W_\psi)^2 \psi(x) d\mu \leq C \int_{B_1} \int_{B_1} (w(x) - w(y))^2 \min\{\psi(x), \psi(y)\} d\nu.$$

Proof. Define $\Psi(x) := \frac{\psi(x)}{|x|^{2\gamma}}$, that is a radial decreasing function. Then using [22, Corollary 6] we get

$$\int_{B_1} (w(x) - \bar{W}_\Psi)^2 \Psi(x) dx \leq C \int_{B_1} \int_{B_1} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} \min\{\Psi(x), \Psi(y)\} dx dy,$$

where

$$\bar{W}_\Psi = \frac{\int_{B_1} w(x) \Psi(x) dx}{\int_{B_1} \Psi(x) dx}.$$

Substituting Ψ by its value, we get

$$\int_{B_1} (w(x) - \bar{W}_\Psi)^2 \Psi(x) dx = \int_{B_1} (w(x) - W_\psi)^2 \psi(x) d\mu,$$

and

$$\int_{B_1} \int_{B_1} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} \min\{\Psi(x), \Psi(y)\} dx dy = \int_{B_1} \int_{B_1} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} \min\left\{\frac{\psi(x)}{|x|^{2\gamma}}, \frac{\psi(y)}{|y|^{2\gamma}}\right\} dx dy.$$

Hence, to finish we just have to show that

$$\min\left\{\frac{\psi(x)}{|x|^{2\gamma}}, \frac{\psi(y)}{|y|^{2\gamma}}\right\} \leq \frac{\min\{\psi(x), \psi(y)\}}{|x|^\gamma |y|^\gamma} \text{ in } B_1 \times B_1.$$

Without loss of generality we can assume that $|x| \geq |y|$.

Define $H(s) := \frac{\psi(s)}{s^{2\gamma}}$, that is a decreasing function in $(0, 1)$. Let $s_1 := |x|$ and $s_2 := |y|$, then

$$\min\left\{\frac{\psi(x)}{|x|^{2\gamma}}, \frac{\psi(y)}{|y|^{2\gamma}}\right\} = H(s_1).$$

Using that ψ is decreasing, we obtain that $\psi(s_1) \leq \psi(s_2)$. Thus

$$\frac{\min\{\psi(x), \psi(y)\}}{|x|^\gamma |y|^\gamma} = \frac{\psi(s_1)}{s_1^\gamma s_2^\gamma}.$$

Since $s_2 \leq s_1 \leq 1$, we conclude that $H(s_1) \leq \frac{\psi(s_1)}{s_1^\gamma s_2^\gamma}$ and the result follows. \square

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